Maximum Multicommodity Flow: We are given a graph $G(V,E)$ with capacities $u(e)$ on the edges, and $k$ pairs of terminals $(s_i, t_i), i = 1, 2, \ldots, k$. The goal is to route flow $d_i$ from $s_i$ to $t_i$, so that $\sum_{i=1}^k d_i$ is maximized. (Note that the sets $\{s_i\}$ and $\{t_i\}$ might not be disjoint.) We will develop a combinatorial algorithm for this problem similar to the algorithm for maximum concurrent flow. If $f(P)$ denotes the flow along path $P$, and $Q_i$ denotes the set of paths from $s_i$ to $t_i$, we can formulate the following positive linear program:

$$\text{Maximize } \sum_{i=1}^k d_i$$

$$\sum_{P \in Q_i} f(P) \geq d_i \quad \forall i$$

$$\sum_{P, e \in P} f(P) \leq u(e) \quad \forall e \in E$$

$$f(P) \geq 0 \quad \forall P \in \bigcup_{i=1}^k Q_i$$

$$d_i \geq 0 \quad \forall i$$

Problem 3-1. Dual Problem: Formulate the dual of this problem. Use the variables $l(e)$ for the edge constraints, and the variables $\text{dist}_i$ for the terminal constraints. Define the volume $D(l)$ of the system as $\sum_e l(e)u(e)$.

1. Show that for the optimal solution, the function $l(e)$ is a metric (in the sense that it is non-negative and, for all $x, y, z \in V$ such that $xy \in E$, $l(xy) \leq c_l(xz) + c_l(zy)$, where $c_l(vw)$ denotes the length of the shortest path from $v$ to $w$ when edge lengths are defined by $l$).

2. Show that for the optimal solution, $\text{dist}_i$ can be set to the shortest path length from $s_i$ to $t_i$ under the metric $l$ without changing the value of the optimum.

3. Given a metric $l$, let $\alpha(l)$ denote the minimum distance between terminal pairs. Show that the dual is effectively minimizing $\frac{D(l)}{\alpha(l)}$ over all length metrics $l$.

4. Suppose the variables $l(e)$ were constrained to be either 0 or 1. In this case, what problem is the dual program solving?

Problem 3-2. Complementary Slackness: Write down the primal and dual complementary slackness constraints. Consider the optimal primal and dual solutions.

1. Show that for $P \in Q_i$, where $Q_i$ is the set of paths from $s_i$ to $t_i$, if $f(P) > 0$, then the length of $P$ in metric $l$ is one.

2. Show that if $l(e) > 0$, then edge $e$ is saturated.

Problem 3-3. The Algorithm: We will solve this problem for the case of unit capacities $u(e) = 1$. The algorithm proceeds in iterations. Let $l_{i-1}$ be the length function at the beginning of the $i^{th}$ iteration, and $f_{i-1}$ denote the flow routed so far. Let $\alpha(i - 1)$ denote the minimum distance between terminals in metric $l_{i-1}$, and $D(i - 1)$ denote the volume of the system. Let $P$ be a path of length
α(i − 1) connecting some terminal pair. We push one unit of flow along P, and for edge e ∈ P, set \( l_i(e) = l_{i-1}(e)(1 + \epsilon) \). We stop at the first time \( t \) such that \( \alpha(t) \geq 1 \).

Essentially, the algorithm finds the path with minimum capacity violation and pushes one unit of flow along it. This path is the shortest path using a length function which is exponential in the violation. Note that \( f_t \) does not satisfy capacity constraints and is therefore infeasible.

Initially, we set \( l_0(e) = \delta \) for all edges. We will choose \( \delta \) later. Let \( \beta \) denote the optimal value of the dual.

Note that \( \alpha(0) \leq \delta n \). Also note that \( f_i = i \).

1. Show that \( D(i) = D(0) + \epsilon \sum_{j=1}^{i} \alpha(j-1) \).
2. Consider the length function \( l_i - l_0 \), and let \( \alpha(l_i - l_0) \) denote the length of the shortest path from any source to the corresponding sink under this length function. Show that \( \beta \leq \frac{D(i)-D(0)}{\alpha(l_i - l_0)} \), and conclude that \( \alpha(i) \leq \delta n + \frac{D(i)-D(0)}{\beta} \).
3. Now show that \( \alpha(i) \leq \delta n(1 + \epsilon/\beta)^i \). Conclude that \( \alpha(i) \leq \delta ne^{\epsilon i/\beta} \).
4. Finally, show that \( f_t = t \geq \frac{\beta \ln(\delta n)^{-1}}{\epsilon} \).

**Problem 3-4. Feasible Flow:** The algorithm described above could easily violate capacities. Note that whenever we route one unit of demand through an edge \( e \), we increase its length by a factor of \( 1 + \epsilon \).

1. Using the fact that \( l_0(e) = \delta \), and \( t \) is the first time instant for which \( \alpha(t) \geq 1 \), show that the total flow through \( e \) is at most \( \log_{1+\epsilon} \frac{1+\epsilon}{\delta} \).
2. Show that \( \frac{f_t}{\log_{1+\epsilon} \frac{1+\epsilon}{\delta}} \) is a feasible flow.

**Problem 3-5. Approximation Ratio:** Let \( \gamma \) denote the ratio between the optimal dual solution and the flow we obtain, that is \( \gamma = \frac{\beta}{f_t} \log_{1+\epsilon} \frac{1+\epsilon}{\delta} \). Show that for \( \delta = (1+\epsilon)((1+\epsilon)n)^{-1/\epsilon} \), \( \gamma \leq (1-\epsilon)^{-2} \).

**Problem 3-6. Running Time:** Show that the algorithm described above computes a \( (1-\epsilon)^{-2} \) approximation to max multicommodity flow in time \( O((\frac{m}{\epsilon^2} \log n)kT_{SP}) \), where \( T_{SP} \) is the time taken to compute single source shortest paths.

**Problem 3-7. Optional:** Suppose we remove the unit capacity assumption. We modify the algorithm as follows. As before, let \( P \) be the shortest path in metric \( l_{i-1} \). Let \( u \) denote the minimum capacity edge along this path. We push \( u \) units of flow along this path, and for edge \( e \) along this path, set \( l_i(e) = l_{i-1}(e)(1 + \epsilon u/u(e)) \). We terminate at the first time \( t \) such that \( \alpha(t) \geq 1 \). The values \( l_0(e) \) are set as before. Note that \( f_i \) is no longer equal to \( i \). Show that after appropriate scaling of the flow and choice of \( \delta \), this algorithm produces a \( (1-\epsilon)^{-2} \) approximation to maximum multicommodity flow.