This is a running list of problems posed in class. At the end of the course you should submit at least half-of them. The problems are numbered by the lecture number and then by the exercise index. For instance, Exercise 3.2 would refer to the second exercise given during the third lecture. Often the exercises arise in the context of the lecture where the appropriate definitions are given.

**Exercise 1.1.** Show that the constraint \( \{ E \text{is a valid tour} \} \) can be expressed using linear constraints for \( x \in \{0, 1\}^n \).

**Exercise 1.2.** Given a weak separation oracle for a convex cone \( K \), show that one can construct a weak separation oracle for \( N(K) \) and \( N_+(K) \).

**Exercise 1.3.** Show that the constraints in (14) are necessary.

**Exercise 2.1.** Show that the constraint \( M_t(y) \succeq 0 \) is satisfied by integral solutions \( y \in \{0, 1\}^{2^n} \) where \( y_I(x) := \prod x_i \) for all \( I \subseteq [n] \) and \( x \in \{0, 1\}^n \).

**Exercise 3.1.** For Sherali Adams with 2 rounds, how high can \( \sum_{i=1}^n y_i \) be for valid solutions \( y \in SA_2(LP) \).

**Exercise 3.2.** Show that \( \sum ||v_i||^2 \leq 1 \) when there exists a unit vector \( v_0 \) such that \( v_i \cdot v_0 = ||v_i||^2 \) for all \( i \in [n] \) and \( v_i \perp v_j \) for all \( i \neq j \in [n] \).

**Exercise 3.3.** Show that if in the basic LP we used the weaker constraints:

\[
\sum_{t \leq t' - 1} x_{it} \geq x_{jt'}, \forall i < j
\]

then for any solution \( y \in \text{Las}_t(K) \) for \( t \geq 3 \) it would be true that:

\[
\sum_{t \leq t' - 1} x_{it} \geq \sum_{t \leq t'} x_{jt'}, \forall i < j
\]

*Hint: use the decomposition theorem on an appropriately chosen set \( S \).*

**Exercise 4.1.** For \( u \in \mathbb{R}^d \) let \( u^\otimes k = u \otimes \ldots \otimes u \) denote the \( k \)-tensor product of \( u \). Show that for all \( u, v \in \mathbb{R}^d \), it holds that \( \langle u^\otimes k, v^\otimes k \rangle = ((u, v))^k \).

**Exercise 4.2.** For a graph \( G = (V, E) \) consider its Laplacian \( L_G = \sum_{(i,j)} (e_i - e_j)(e_i - e_j)^T \). Show that \( \lambda_{\text{max}}(L_G) \frac{n}{2} - f_G \) has degree-2 sos certificate where \( f_G \) is the max cut polynomial.

**Exercise 4.3.** For \( f : \{0, 1\}^n \to \mathbb{R} \) with degree at most \( d \) for even \( d \in \mathbb{N} \) there exists \( M \in \mathbb{R}_{\geq 0} \) such that \( M - f \) has degree-\( d \) sos certificate. Also \( M \) can be chosen \( n^{O(d)} \) times the largest coefficient of \( f \) in the monomial basis.
Exercise 4.4. If \( \mu : \{0, 1\}^n \to \mathbb{R} \) has degree \( > \ell \) what is the projection of \( \mu \) onto \( U_\ell \)? where \( U_\ell \) denotes the linear span of degree-\( \ell \) multilinear polynomials.

Exercise 4.5. Show that if \( \mu \) is a degree-2n pseudo-distribution, then \( \mu(x) \geq 0 \) for all \( x \in \{0, 1\}^n \).

Exercise 4.6. The following two statements are equivalent:

1. \( \mu \) is a pseudo-distribution.
2. \( \hat{E}_\mu 1 = 1 \) and \( \hat{E}_\mu \left\{ (1, x)^{\otimes d/2} \left[ (1, x)^{\otimes d/2} \right]^\top \right\} \succeq 0. \)

Exercise 4.7. For all \( d \geq 0 \) and for any pseudo-distribution \( \mu \) of degree \( d \), there exists another pseudo-distribution \( \mu' \) with the same pseudo-moments up to degree \( d \) and \( |\mu'(x)| \leq 2^{-n} \sum_{d'=0}^{d} \binom{n}{d'} \).

Exercise 4.8. For degree \( d \) pseudo-distributions over \( \{0, 1\}^n \) there exists a separation algorithm with running time \( n^{O(d)} \).

Exercise 4.9. Show that for every \( d \in \mathbb{N} \), the following set of pseudo moments admits a separation algorithm with running time \( n^{O(d)} \),

\[ \mathcal{M}_d = \left\{ \hat{E}_\mu (1, x)^{\otimes d} \left| \mu \text{ is deg-} d \text{ pseudo distribution over } \{0, 1\}^n \right. \right\}. \]

Exercise 5.1. Prove that Cheeger’s inequality \( \Rightarrow \) degree 2 SoS certificate for \( f_G(x) - \frac{1}{2} \phi^2(G) \cdot \frac{d}{n} |x|(n - |x|) \).

Hint: Note the quadratic form of Normalized laplacian can be represented in terms of \( f_G(x) \).

\[ \langle x, L_G x \rangle = \frac{1}{d} f_G(x) \]

Exercise 7.1. Let \( v \) be the max eigenvector of \( M \). Show that w.h.p, \( \max_i \langle a_i, v \rangle^2 \geq 1 - o(1) \) and every \( a_i \) is still maximal w.p. \( n^{-O(1)} \).

Exercise 7.2. Suppose we instead defined \( T = \sum_{i=1}^r a_i^{\otimes 3} + R \) where entries of \( R \) are IID Gaussian. How large can \( R \) be for the problem to still be well defined? Explain why Jenrich’s algorithm will fail.

Exercise 7.3. Show that \( \|T\|_{sos_k} \) is a norm.

Exercise 7.4. Show that the 2-norm of a \( k \)-tensor (that is, its 2-norm as a large vector in \( \mathbb{R}^{n^k} \)) is an upper bound on its sos \( d \) norm, for any \( d \geq k \).

Exercise 7.5. Show that if \( k \) is even, the norm \( \|T\|_{op} \) given by unfolding \( T \) to a \( n^{k/2} \times n^{k/2} \) matrix and measuring its spectral norm satisfies \( \|T\|_{op} \geq \|T\|_{sos_k} \).

Exercise 7.6. What is the analogue of the sos norm for matrices? Prove that they collapse to \( \|M\|_{inv} \).

Exercise 7.7. Show that \( \left( \hat{E} \sum_{i=1}^r \langle a_i, x \rangle^2 \right)^{1/2} \leq 1. \)
Exercise 7.8. Show that the optimization problem:

\[
\begin{align*}
\text{arg max} & \quad \tilde{E}(T, x^\otimes 3) \\
\text{s.t} & \quad \text{deg}\tilde{E} = 6 \text{ satisfies } \{\|x\|^2 = 1\} \\
& \quad \|\tilde{E}x^T\|_{op} \leq \frac{1}{r} \\
& \quad \|\tilde{E}(x \otimes x)(x \otimes x)^T\|_{op} \leq \frac{1}{r}
\end{align*}
\]

is a convex program.

Exercise 7.9. Show that \[\|\sum_{i \leq n} M_{ij}M_{ij}^T\|^{1/2} \leq \frac{1}{r}.\]

Exercise 8.1. Provide an argument for why the information theoretic threshold should grow as \(n \geq \Omega\left(\frac{k \log p}{\lambda^2}\right)\) (where you can think of \(\lambda\) as a small constant and ignore it) and why does it have to grow with the \(\log p\) of the dimension?

Exercise 8.2. Show that the way \(x^*\) is defined \((b_i^j \oplus b_j^i \oplus b_k^j = x_i^* \oplus x_j^* \oplus x_k^* \oplus 1)\), it will be a satisfying assignment.

Exercise 8.3. Prove the expansion property for random 3-CSPs.

Exercise 8.4. Prove that the last equality \[\sum_i \tilde{E}(q_i^2) + \sum_{i \neq j} \tilde{E}(q_iq_j) = \sum_i \tilde{E}(q_i^2)\] is true.

References