This is a running list of problems posed in class. At the end of the course you should submit at least half of them. The problems are numbered by the lecture number and then by the exercise index. For instance, Exercise 3.2 would refer to the second exercise given during the third lecture. Often the exercises arise in the context of the lecture where the appropriate definitions are given.

**Exercise 1.1.** Show that the constraint \( \{ E_x \text{ is a valid tour} \} \) can be expressed using linear constraints for \( x \in \{0, 1\}^n \).

**Exercise 1.2.** Given a weak separation oracle for a convex cone \( K \), show that one can construct a weak separation oracle for \( N(K) \) and \( N_+(K) \).

**Exercise 1.3.** Show that the constraints in (14) are necessary.

**Exercise 2.1.** Show that the constraint \( M_t(y) \preceq 0 \) is satisfied by integral solutions \( y \in \{0, 1\}^{2n} \) where \( y_{I}(x) := \prod_{i \in I} x_i \) for all \( I \subseteq [n] \) and \( x \in \{0, 1\}^n \).

**Exercise 3.1.** For Sherali Adams with 2 rounds, how high can \( \sum_{i=1}^{n} y_i \) be for valid solutions \( y \in SA_2(LP) \).

**Exercise 3.2.** Show that \( \sum_{i} ||v_i||^2 \leq 1 \) when there exists a unit vector \( v_0 \) such that \( v_i \cdot v_0 = ||v_i||^2 \) for all \( i \in [n] \) and \( v_i \perp v_j \) for all \( i \neq j \in [n] \).

**Exercise 3.3.** Show that if in the basic LP we used the weaker constraints:

\[
\sum_{t \leq t' - 1} x_{it} \geq x_{jt'}, \forall i < j
\]

then for any solution \( y \in \text{Las}_t(K) \) for \( t \geq 3 \) it would be true that:

\[
\sum_{t \leq t' - 1} x_{it} \geq \sum_{t \leq t'} x_{jt'}, \forall i < j
\]

*Hint: use the decomposition theorem on an appropriately chosen set \( S \).*

**Exercise 4.1.** For \( u \in \mathbb{R}^d \) let \( u \otimes_k = u \otimes \ldots \otimes u \) denote the \( k \)-tensor product of \( u \). Show that for all \( u, v \in \mathbb{R}^d \), it holds that \( \langle u \otimes_k , v \otimes_k \rangle = \langle (u, v) \rangle^k \).

**Exercise 4.2.** For a graph \( G = (V, E) \) consider its Laplacian \( L_G = \sum_{(i,j)} (e_i - e_j)(e_i - e_j)^T \). Show that \( \lambda_{\max}(L_G) \frac{n}{2} - f_G \) has degree-2 sos certificate where \( f_G \) is the max cut polynomial.

**Exercise 4.3.** \( \forall f : \{0, 1\}^n \to \mathbb{R} \) with degree at most \( d \) for even \( d \in \mathbb{N} \) there exists \( M \in \mathbb{R}_{\geq 0} \) such that \( M - f \) has degree-\( d \) sos certificate. Also \( M \) can be chosen \( n^{O(d)} \) times the largest coefficient of \( f \) in the monomial basis.
Exercise 4.4. If $\mu : \{0,1\}^n \to \mathbb{R}$ has degree $> \ell$ what is the projection of $\mu$ onto $U_\ell$? Where $U_\ell$ denotes the linear span of degree-$\ell$ multilinear polynomials.

Exercise 4.5. Show that if $\mu$ is a degree-2n pseudo-distribution, then $\mu(x) \geq 0$ for all $x \in \{0,1\}^n$.

Exercise 4.6. The following two statements are equivalent:

1. $\mu$ is a pseudo-distribution.
2. $\bar{E}_\mu 1 = 1$ and $\bar{E}_\mu \left\langle (1, x)^{\otimes d/2} \left[ (1, x)^{\otimes d/2} \right]^\top \right\rangle \succeq 0$.

Exercise 4.7. For all $d \geq 0$ and for any pseudo-distribution $\mu$ of degree $d$, there exists another pseudo-distribution $\mu'$ with the same pseudo-moments up to degree $d$ and $|\mu'(x)| \leq 2^{-n} \sum_{d' = 0}^{d} \left( \begin{array}{c} n \\ d' \end{array} \right)$.

Exercise 4.8. For degree $d$ pseudo-distributions over $\{0,1\}^n$ there exists a separation algorithm with running time $n^{O(d)}$.

Exercise 4.9. Show that for every $d \in \mathbb{N}$, the following set of pseudo moments admits a separation algorithm with running time $n^{O(d)}$,

$$\mathcal{M}_d = \left\{ \bar{E}_\mu (1, x)^{\otimes d} \mid \mu \text{ is deg-$d$ pseudo distribution over } \{0,1\}^n \right\}.$$

References