1. (a) Prove that every $\ell_2$ metric on $n$ points embeds isometrically into $\ell_2$ with $n-1$ dimensions.

(b) Prove that every $\ell_1$ metric on $n$ points embeds isometrically into $\ell_1$ with $\binom{n}{2}$ dimensions.

<Insert Hint Here>

2. Show that every embedding of the shortest path metric of a cycle graph of length $n$ into the line $\mathbb{R}^1$ with the metric $d(x, y) = |x - y|$ has distortion at least $\Omega(n)$.

<Insert Hint Here>

3. For all finite $p \geq 1$, show that the mapping given in the proof of Bourgain’s theorem is an embedding into $\ell_p$ with distortion $O\left(\log n \frac{p}{\log p}\right)$.

<Insert Hint Here>

4. Prove that the integrality gap of the LP for generalized sparsest cut is exactly equal to the worst-case distortion needed to map $n$-point metrics into $\ell_1$.

<Insert Hint Here>

5. In this problem, you will show that linear projections perform very poorly for dimension reduction in $\ell_1$ (which is in contrast to $\ell_2$). More specifically, we will specify here an explicit set of $O(n)$ points in $\ell_1^n$, and show that any linear embedding of that point set (with $\ell_1$) into $\ell_1^d$ incurs distortion at least $\sqrt{n/d}$.

The point set consists of the origin $O$, the $n$ standard basis vectors $P_i$ (where $P_i$ has 1 in the $i$th coordinate and 0’s elsewhere), and $m = O(n)$ points $Q_i$ with the following property: For every pair of coordinates $j_1, j_2$ and pair of values in $(x_1, x_2) \in \{1, -1\}^2$, exactly $m/4$ of the points $Q_i$ have the coordinates $j_1$ and $j_2$ set to $x_1$ and $x_2$, respectively. (Such set of points $Q_i$ can constructed from the support of a pairwise independent distribution.)

Without loss of generality, consider a linear map $f : \ell_1^n \to \ell_1^d$ that is non-expanding with distortion $\alpha$, i.e. $\forall x, y \in \{O, P_1, \ldots, P_n, Q_1, \ldots, Q_m\}$,

$$\frac{1}{\alpha} \|x - y\|_1 \leq \|f(x) - f(y)\|_1 \leq \|x - y\|_1.$$

Our goal in this problem is to show that $\alpha \geq \sqrt{n/d}$. W.l.o.g., $f$ maps the origin in $\mathbb{R}^n$ to the origin in $\mathbb{R}^d$. 

1
Let \( \sigma_1, \ldots, \sigma_d : \mathbb{R}^n \to \mathbb{R}^1 \) such that \( f = (\sigma_1, \ldots, \sigma_d) \). Consider \( \sigma \in \{\sigma_1, \ldots, \sigma_d\} \) with 
\[
\sigma(x_1, \ldots, x_n) = \sum_{j=1}^{n} \gamma_j x_j.
\]

(a) Prove that \( |\gamma_j| \leq 1 \).

(b) Prove that 
\[
\frac{1}{m} \sum_{i=1}^{m} |\sigma(Q_i)| \leq \sqrt{\sum_{j=1}^{n} \gamma_j^2} \leq \sqrt{\sum_{j=1}^{n} |\gamma_j|}.
\]

(c) Show that 
\[
\frac{n}{d} + \sum_{j=1}^{n} |\sigma(P_j)| \geq 2 \sqrt{\frac{n}{d}} \cdot \frac{1}{m} \sum_{i=1}^{m} |\sigma(Q_i)|,
\]
and conclude that 
\[
n + \sum_{j=1}^{n} \|f(P_j)\|_1 \geq 2 \sqrt{\frac{n}{d}} \cdot \frac{1}{m} \sum_{i=1}^{m} \|f(Q_i)\|_1.
\]

(d) Show that the distortion \( \alpha \geq \sqrt{\frac{n}{d}} \).