1 Introduction

This lecture covers some basic embedding results for embedding metric spaces into $\ell_\infty$. These ideas are used in other embedding results, e.g., Bourgain’s theorem which provides embeddings into $\ell_2$.

2 Embeddings into $\ell_\infty$

We begin by showing that any finite metric embeds isometrically into $\ell_\infty$.

**Theorem 2.1.** Every $n$-point metric space $(V, d)$ embeds isometrically into $\ell_\infty$.

**Proof.** Let $V = \{v_1, \ldots, v_n\}$. We will map each point $v \in V$ to a vector of length $n$, where the $i$th coordinate is the distance between $v$ and $v_i$. Formally, define $f_i(v_j) = d(v_i, v_j)$, and let

$$f(v_j) = \left( \begin{array}{c} f_1(v_j) \\ \vdots \\ f_n(v_j) \end{array} \right).$$

**Claim 2.2.** For any $v_i, v_j$, we have $\|f(v_i) - f(v_j)\|_\infty \geq d(v_i, v_j)$.

**Proof.**

$$\|f(v_i) - f(v_j)\|_\infty = \max_k |f_k(v_i) - f_k(v_j)| \geq |f_i(v_i) - f_i(v_j)| = d(v_i, v_j).$$

**Claim 2.3.** For any $v_i, v_j$, we have $\|f(v_i) - f(v_j)\|_\infty \leq d(v_i, v_j)$.

**Proof.** By the triangle inequality, for any $k$,

$$|f_k(v_i) - f_k(v_j)| = |d(v_i, v_k) - d(v_k, v_j)| \leq d(v_i, v_j).$$

The theorem follows from the two claims.

The embedding in the proof used $n$ coordinates. Do we need to all $n$ coordinates to construct an isometric embedding?

The answer is no: $n - 1$ coordinates suffice. If we remove any one coordinate, it is easy to check that the first claim still holds (plugging in $j$ instead of $i$ if necessary). Eliminating a coordinate cannot increase the $\ell_\infty$ norm, so the second claim also holds.

However, we cannot use less than $n - 1$ coordinates without allowing some distortion.

If we do allow for some distortion, we can drastically reduce the number of coordinate. The following theorem yields an upper bound for the number of coordinates necessary to achieve distortion $D$. 


**Theorem 2.4 (Upper Bound).** Let $D = 2q - 1 \geq 3$ be an odd integer. Given an $n$-point metric space $(V,d)$, there exists an embedding with distortion $D$ of $(V,d)$ into $\ell^k_\infty$ with $k = O(qn^{1/q} \ln n)$.

For example, for distortion $D = 3$, we get an embedding into $k = O(\sqrt{n} \ln n)$ dimensions. For a smaller value of $D$, we cannot find an embedding with a number of dimensions that is asymptotically better than needed for the isometric embedding described above.

**Theorem 2.5 (Lower Bound).** There is an $n$-point metric space $(V,d)$ such that every embedding of $(V,d)$ into $\ell_\infty$ with distortion $D < 3$ requires $\Omega(n)$ dimensions.

For $D = 3$, there is a lower bound of $\Omega(\sqrt{n})$ dimensions, and there are similar matching lower bounds for $D = 5, 7, 11$ (and maybe more).

We will prove the upper bound in Theorem 2.4 using a mapping called a Fréchet embedding.

**Definition 2.6.** A Fréchet embedding is a map $f : (V,d) \rightarrow \ell^k_p$ which consists of $k$ functions $f_i$ (one per coordinate), each of which has the form $f_i(v) = d(v, A_i)$, where $A_i \subset V$. The distance $d(v,A)$ is the minimum distance between a point $a \in A$ and the point $v$.

More generally, we will sometimes allow scaling by constants $\alpha_i$ in the definition of Fréchet embeddings, that is, each coordinate will be given by $f_i(v) = \alpha_i d(v, A_i)$.

**Fact 2.7.** Fréchet embeddings (without scaling) are non-expanding for a single coordinate, that is, for every $i$,

$$|f_i(u) - f_i(v)| \leq d(u,v).$$

This property of Fréchet embeddings follows from the triangle inequality. Since for every $u,v$, $d(u,A) \leq d(u,v) + d(v,A)$ (convince yourself why), we get that $|f_i(u) - f_i(v)| = |d(u,A_i) - d(v,A_i)| \leq d(u,v)$.

Note that the isometric embedding into $\ell_\infty$ that we saw earlier today is a Fréchet embedding.

We are now ready to prove Theorem 2.4.

**Proof of Theorem 2.4.** Since we will use a Fréchet embedding and the target metric is $\ell_\infty$, the embedding will be non-expanding ($\|f(u) - f(v)\|_\infty = \max_i |f_i(u) - f_i(v)| \leq d(u,v)$). Hence in order to prove that the embedding has distortion $D$, it suffices to show that for some coordinate $i$, we have $|f_i(u) - f_i(v)| \geq \frac{1}{D} d(u,v)$.

Recall that for a point $v \in V$ and radius $r \in \mathbb{R}^\geq$, the (closed) ball $B(v,r)$ around $v$ of radius $r$ contains all the points in the metric space that are within distance $r$ from $v$, that is, $B(v,r) = \{u \in V \mid d(u,v) \leq r\}$. The open ball is defined similarly as $B^0(v,r) = \{u \in V \mid d(u,v) < r\}$.

The intuition behind the proof is as follows. To get a difference $|f_i(u) - f_i(v)| \geq \Delta$ for coordinate $i$, we need to have some $r$ such that $B^0(u,r+\Delta) \cap A_i = \emptyset$, but $B(v,r) \cap A_i \neq \emptyset$ or vice versa (replace the roles of $u$ and $v$). This would imply that $d(u,A_i) \geq r+\Delta$ while $d(v,A_i) \leq r$ (or the same with the opposite roles). If we can find $r$ such that the number of points in $B^0(u,r+\Delta)$ is not too large compared to $B(v,r)$, we can pick $A_i$ at random and hope for it to intersect $B(v,r)$ but not $B^0(u,r+\Delta)$ (see Figure 1).

The embedding of $(V,d)$ into $\ell^k_\infty$ will be a Fréchet embedding constructed in the following way:

- Let $p = n^{-1/q}$.
- For $j = 1, \ldots, q$, let $p_j = \min\{1/2, p^j\}$.
Let $m = \lceil 24n^{1/q} \ln n \rceil$.

Let $k = mq$ be the number of coordinates in our embedding.

For each $i \in [m]$, $j \in [q]$, let $A_{ij}$ be a random sample of points in $V$ with density $p_j$, i.e., each point $v \in V$ is chosen to be in $A_{ij}$ independently with probability $p_j$. Our mapping is the Fréchet embedding with these $k$ sets.

We proceed to show that this randomized construction returns an embedding with distortion $D$ with probability at least $1/2$. We say that a subset $A_{ij}$ is good for $(u, v)$ if $|d(u, A_{ij}) - d(v, A_{ij})| \geq \frac{1}{D}d(u, v)$.

**Lemma 2.8.** For any pair $u, v \in V$, there exists some $j$ such that if $A_{ij}$ is a random sample of points from $V$ where each point is sampled with probability $p_j$, then

$$\Pr[A_{ij} \text{ is good for } (u, v)] \geq \frac{p}{12}.$$

Before we prove the lemma, let us first show how it can be used to complete the proof of the theorem. For any pair $u, v \in V$,

$$\Pr[\text{no } A_{ij} \text{ is good for } (u, v)] \leq \left(1 - \frac{p}{12}\right)^m \leq e^{-\frac{pm}{12}} \leq \frac{1}{n^2}.$$

By the union bound, since there are $\binom{n}{2}$ pairs $(u, v)$, the probability that there exists a pair with no good $A_{ij}$ is less than $\frac{1}{n^2} \binom{n}{2} \leq 1/2 < 1$. Hence, there exists some choice of sets $A_{ij}$ for which the mapping has distortion $D$. The probability of success (that is, choosing an embedding with distortion $D$) can be amplified by choosing larger $m$.

Finally, we prove Lemma 2.8. Recall that we want to find a coordinate with distance at least
$d(u, v)/D$. Define $Δ = d(u, v)/D$. Consider the following sequence of balls:

$$
\begin{align*}
B_0 &= B(u, 0) \\
B_1 &= B(v, Δ) \\
B_2 &= B(u, 2Δ) \\
\vdots \\
B_q &= \begin{cases} 
B(u, qΔ) & \text{if } q \text{ is even} \\
B(v, qΔ) & \text{if } q \text{ is odd}
\end{cases}
\end{align*}
$$

Our goal is to find two consecutive balls $B_t$ and $B_{t+1}$ such that the number of points in $B_{t+1}$ is not much greater than the number in $B_t$. For each $t$, define $n_t = |B_t|$, the number of points in $B_t$. Recall that $D = 2q - 1$, and note that for any $t \leq q - 1$,

$$B_t \cap B^o_{t+1} = \emptyset,$$

because $(2t + 1)Δ \leq (2q - 1)Δ = d(u, v)$. (Here $B^o_{t+1}$ denotes the corresponding open ball.)

**Claim 2.9.** There exist some $t < q$ and $j \leq q$ such that $n_t \geq n^{\frac{j-1}{α}}$ and $n_{t+1} \leq n^{\frac{j}{α}}$.

**Proof.** If there is some $t < q$ such that $n_t \geq n_{t+1}$, the claim holds for that $t$ and the maximum $j$ such that $n_t \geq n^{\frac{j-1}{α}}$. Otherwise, $n_0 < n_1 < \cdots < n_t$. Consider the $q$ intervals $\left[\frac{t}{α} + 1, n^{\frac{j}{α}}\right]$ for $j = 1, \ldots, q$. Since there are $q + 1$ increasing values $n_0, \ldots, n_q$, by the pigeonhole principle, there must be some $t$ and $j$ such that $n_t$ and $n_{t+1}$ belong to the same interval $\left[\frac{t}{α} + 1, n^{\frac{j}{α}}\right]$. The claim holds for those $t$ and $j$. \hfill \Box

We are now ready to show that for the values $t$ and $j$ that were found in Claim 2.9, with probability at least $p/12$, the set $A_{ij}$ intersects $B_t$ but not $B^o_{t+1}$. As explained earlier, that would imply that $A_{ij}$ is good for $(u, v)$ (note that the difference in radius between $B_t$ and $B^o_{t+1}$ is $Δ$). Let $E_1$ be the event that $A_{ij} \cap B_t \neq \emptyset$. Let $E_2$ be the event that $A_{ij} \cap B^o_{t+1} = \emptyset$. Because each point in $A_{ij}$ is chosen independently at random with probability $p_j$, we have

$$\Pr[E_1] = 1 - \Pr[A_{ij} \cap B_t = \emptyset] = 1 - (1 - p_j)n_t \geq 1 - e^{-p_jn_t}.$$ 

Plugging in $p_j = \min\{1/2, n^{-\frac{1}{4}}\}$ and $n_t \geq n^{\frac{j-1}{α}}$, we have

$$\Pr[E_1] \geq 1 - e^{-n_t \min\{1/2, n^{-\frac{1}{4}}\}} \geq \min\{1 - e^{-\frac{1}{2}n^{\frac{j-1}{α}}}, 1 - e^{-p}\} \geq \min\{1 - e^{-\frac{1}{2}}, 1 - e^{-p}\},$$

where the last inequality follows because $n^{\frac{j-1}{α}} \geq 1$. If $p \geq 1/2$, then this minimum is $1 - e^{-\frac{1}{2}} > 1/3 \geq p/3$. If $p < 1/2$, then using the first three terms of the Taylor expansion for $e^x$, we get $1 - e^{-p} \geq p - p^2/2 > p/3$. We conclude that in all cases,

$$\Pr[E_1] \geq p/3.$$ \hfill (1)

Now,

$$\Pr[E_2] \geq (1 - p_j)^{n_{t+1}} \geq (1 - p_j)^{n^{\frac{j}{α}}}.$$
We again split this into two cases based on the value of $p_j$. If $p_j > 1/2$, then $n^{\frac{1}{q}} = 1/p_j$, and hence

$$(1 - p_j) ^ {n^{\frac{1}{q}}} = (1 - p_j)^{1/p_j} \geq 1/4,$$

where the last inequality holds by checking that $(1 - p_j)^{\frac{1}{p_j}}$ is decreasing in $p_j$ and is hence minimized when $p_j = 1/2$.

If $p_j = 1/2$, then we must have $n^{\frac{1}{q}} \leq 2$, and

$$(1 - p_j) ^ {n^{\frac{1}{q}}} \geq \left( \frac{1}{2} \right)^2 = 1/4.$$ 

In all cases, we have

$$\Pr[E_2] \geq 1/4. \tag{2}$$

It follows from Equation (1) and Equation (2) that

$$\Pr[E_1 \cap E_2] = \Pr[E_1] \cdot \Pr[E_2] \geq p/12,$$

where we used the fact that $B_t$ and $B_{t+1}$ are disjoint, and as a result, the events $E_1$ and $E_2$ are independent. This completes the proof of Lemma 2.8.

3 Embeddings into $\ell_2$

In the previous lecture, we mentioned Bourgain’s theorem.

**Theorem 3.1** (Bourgain [1]). Any $n$-point metric embeds into $\ell_2$ with distortion $O(\log n)$.

We claim that the proof of Theorem 2.4 already provides an embedding of any $n$-point metric into $\ell_2$ with distortion $O(\log^2 n)$.

**Proposition 3.2.** Any $n$-point metric embeds into $\ell_2$ with distortion $O(\log^2 n)$.

**Proof.** Let $q = \log n$, and consider the mapping provided by Theorem 2.4. This mapping is an embedding into $\ell_\infty$ with dimension $k = O(\log n \cdot n^{\frac{1}{\log n}} \log n) = O(\log^2 n)$ and distortion $O(\log n)$. We use this mapping (which we denote by $f$), and then embed these vectors into $\ell_2^k$ using the identity map, that is, we consider the same points but with $\ell_2$ distance instead of $\ell_\infty$.

Consider two points $u, v$ in the original metric space. We have shown in the proof of Theorem 2.4 that

$$\frac{1}{2q-1} d(u, v) \leq \|f(u) - f(v)\|_\infty \leq d(u, v).$$

Now, for every $x \in \mathbb{R}^k$, $\|x\|_2 \leq \sqrt{k} \|x\|_\infty$ (convince yourself why). Together, we get that the embedding into $\ell_2$ satisfies

$$\frac{1}{2q-1} d(u, v) \leq \|f(u) - f(v)\|_2 \leq \sqrt{k} d(u, v),$$

and since $k = O(\log^2 n)$ and $2q - 1 = O(\log n)$, we conclude that $f$ is an embedding into $\ell_2$ with distortion $O(\log^2 n)$. \qed
3.1 Overview of the Proof of Bourgain’s Theorem

We will prove Bourgain’s theorem in the next lecture, but we provide a brief sketch of the analysis here.

As in Theorem 2.4, we will construct a Fréchet embedding with random sets of varying densities. To perform the analysis, for each pair \((u, v)\) we will grow balls around \(u\) and \(v\) with radii in \([0, d(u, v)/2]\). Let \(q = \lfloor \log_2 n \rfloor + 1\) and \(r_1, r_2, \ldots, r_q\) be radii such that \(r_j\) is the smallest choice of radius such that both \(|B(u, r_j)|, |B(v, r_j)| \geq 2^j\) (if this radius is greater than \(d(u, v)/2\), we set \(r_j = d(u, v)/2\)). We define \(r_0 = 0\) and \(\Delta_j = r_j - r_{j-1}\) so that \(\sum_{j=1}^q \Delta_j \geq d(u, v)/2\).

The following claim forms the core of the proof.

**Claim 3.3.** If a random set \(A_j\) is sampled from \(V\) such that each point is included independently with probability \(\frac{1}{2^j}\), then

\[
\Pr[|d(u, A_j) - d(v, A_j)| \geq \Delta_j] \geq \frac{1}{12}.
\]

Assuming this claim, we can show using the Cauchy-Schwarz inequality that if enough sets are sampled with each of the densities \(2^{-j}\), then with constant probability, the embedding into \(\ell_2\) does not contract the distance between any two points too much.

**References**