CS 369M: Metric Embeddings and Algorithmic Applications October 2, 2018

Lecture 3: Bourgain's Theorem and the Sparsest Cut Problem

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1 Introduction

In this lecture, we show the proof of Bourgain's theorem, which claims that every n-point metric space embeds into ℓ_2 with distortion $O(\log n)$. Then, we introduce the sparsest cut problem, for which we will later show an approximation algorithm that builds on Bourgain's theorem.

2 Bourgain's Theorem

In the previous lecture, we proved the following theorem.

Theorem 2.1. *Let $D = 2q - 1 \geq 3$ be an odd integer and let (V, d_V) be an n-points metric space (i.e., $|V| = n$). Then there is a D-embedding of (V, d_V) into ℓ_∞^k with $k = O(qn^{1/q} \ln n)$.*

In this lecture, we prove Bourgain's theorem, stated as follows.

Theorem 2.2 (Bourgain [1])**.** *Every n-point metric space (V, d_V) can be embedded in ℓ_2 with distortion at most $O(\log n)$.*

The proof will be similar to the proof of Theorem 2.1. For the proof of Bourgain's theorem, refer to Matoušek's lecture notes, Section 4.2, pages 107-110 [2].

The embedding in the proof of Bourgain's theorem is mapping into ℓ_2 with $k = O(\log^2 n)$ dimensions. In ℓ_2, we can reduce the dimension to $O(\log n)$ using the *Johnson-Lindenstrauss lemma.*

Theorem 2.3 (Johnson-Lindenstrauss Lemma)**.** *Let $0 < \varepsilon < 1$ and consider an n-point subset of $\mathbb{R}^d$. There is a mapping $f : \mathbb{R}^d \to \mathbb{R}^k$ where $k = O\left(\frac{\log n}{\varepsilon^2}\right)$ such that for every two points x, y in the set,*

$$(1 - \varepsilon)\|x - y\|_2 \leq \|f(x) - f(y)\|_2 \leq (1 + \varepsilon)\|x - y\|_2.$$

Furthermore, even in ℓ_1, the dimension of the embedding can be improved to be $O(\log n)$.

Exercise 1. For all finite $p \geq 1$, show that the mapping given in the proof of Bourgain's theorem is an embedding into ℓ_p with distortion $O\left(\frac{\log n}{p}\right)$.

3 The Sparsest Cut Problem

We now introduce the sparsest cut problem, for which we will show an approximation algorithm in the next lecture. Given a graph $G = (V, E)$, for any $S \subseteq V$, consider the cut $(S, \bar{S})$. Denote by $E(S, \bar{S})$ the set of edges whose one endpoint is in S and the other is in $\bar{S}$. For any non-empty

$S \subset V$, we define the *sparsity* of a cut $(S, \bar{S})$ as $\phi(G, S) = \dfrac{|E(S, \bar{S})|}{|S||\bar{S}|}$. In the *uniform sparsest cut* problem, we would like to find the cut with minimum sparsity, that is, compute

$$\phi(G) = \min_{S \subset V : S \neq \emptyset} \phi(G, S) = \min_{S \subset V : S \neq \emptyset} \frac{|E(S, \bar{S})|}{|S||\bar{S}|}.$$

Solving this problem exactly is known to be NP-hard, and assuming the Unique Games Conjecture, a generalized version of this problem is even hard to approximate within any constant factor. Formally, when we are talking about approximation algorithms, we say that an algorithm computes an α-approximate solution to the uniform sparsest cut problem if it returns a cut S such that $\phi(G, S) \leq \alpha \phi(G)$. We usually the note the value of the optimal solution (in this case, $\phi(G)$) by OPT.

The definition of the problem is combinatorial and does not seem related to metrics. We reformulate the problem to be finding a minimum over cut metrics.

Definition 3.1. A metric space (V, d) is a *cut metric* if it embeds isometrically into $\{0, 1\}$ (in $\mathbb{R}^1$) with the ℓ_1 distance. Equivalently, (V, d) is a cut metric if there is a subset $S \subseteq V$ such that

$$d(u, v) = \begin{cases} 0 & u, v \in S \text{ or } u, v \in \bar{S} \\ 1 & \text{otherwise} \end{cases}.$$

We now claim that the sparsest cut problem can be formulated as

$$\phi(G) = \min_{d \text{ is a cut metric}} \frac{\sum_{\{u,v\} \in E} d(u, v)}{\sum_{u,v \in V} d(u, v)}.$$

The cut metrics that we minimize over are defined on V (the vertices of the graph). This representation follows since for the cut metric d that corresponds to the cut $(S, \bar{S})$, $\displaystyle\sum_{\{u,v\} \in E} d(u, v) = |E(S, \bar{S})|$ and $\displaystyle\sum_{u,v \in V} d(u, v) = |S||\bar{S}|$.

We can take this one step further, and instead of optimizing over cut metrics, we can optimize over non-negative linear combination of cut metrics. Non-negative linear combinations of cut metrics are metrics of the form $d(u, v) = \displaystyle\sum_{S} \alpha_S d_S(u, v)$ where for every possible cut S, we have $\alpha_s \geq 0$. When we minimize over non-positive combinations of cut metrics, there is always a cut metric that achieves the minimum, due to the inequality

$$\frac{a_1 + a_2 + \ldots + a_k}{b_1 + b_2 + \ldots + b_k} \geq \min_{i=1,\ldots,k} \frac{a_i}{b_i}.$$

In conclusion, we showed that the uniform sparsest cut problem can be formulated as a problem of minimizing over metrics that are non-negative linear combinations of cut metrics. In the next lecture, we will show an algorithm that solves a *relaxation* of this problem: We will minimize over all metrics, and then use Bourgain's theorem to embed the solution into ℓ_1, which is a non-negative linear combination of cut metrics. This way, we will get an approximate solution to the sparsest cut problem.

References

[1] Jean Bourgain. On Lipschitz embedding of finite metric spaces in Hilbert space. *Israel Journal of Mathematics*, 52(1):46–52, Mar 1985.

[2] Jiří Matoušek. Lecture notes on metric embeddings. `https://kam.mff.cuni.cz/~matousek/ba-a4.pdf`, 2013.