Mathematics

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Outline

Algebra

Number Theory

Combinatorics

Geometry

Sum of Powers

$$
\begin{aligned}
\sum_{k=1}^{n} k^2 &= \frac{1}{6} n(n+1)(2n+1) \\
\sum k^3 &= \left(\sum k\right)^2 = \left(\frac{1}{2} n(n+1)\right)^2
\end{aligned}
$$

- Pretty useful in many random situations
- Memorize above!

Fast Exponentiation

- Recursive computation of a^n:

$$a^n = \begin{cases} 1 & n = 0 \\ a & n = 1 \\ (a^{n/2})^2 & n \text{ is even} \\ a(a^{(n-1)/2})^2 & n \text{ is odd} \end{cases}$$

Implementation (recursive)

```
double pow(double a, int n) {
    if(n == 0) return 1;
    if(n == 1) return a;
    double t = pow(a, n/2);
    return t * t * pow(a, n%2);
}
```

- Running time: $O(\log n)$

Implementation (non-recursive)

```
double pow(double a, int n) {
    double ret = 1;
    while(n) {
        if(n%2 == 1) ret *= a;
        a *= a; n /= 2;
    }
    return ret;
}
```

- You should understand how it works

Linear Algebra

- Solve a system of linear equations
- Invert a matrix
- Find the rank of a matrix
- Compute the determinant of a matrix
- All of the above can be done with Gaussian elimination

Outline

Greatest Common Divisor (GCD)

- $\gcd(a, b)$: greatest integer divides both a and b
- Used very frequently in number theoretical problems
- Some facts:
  - $\gcd(a, b) = \gcd(a, b - a)$
  - $\gcd(a, 0) = a$
  - $\gcd(a, b)$ is the smallest positive number in $\{ax + by \mid x, y \in \mathbf{Z}\}$

Euclidean Algorithm

- Repeated use of $\gcd(a, b) = \gcd(a, b - a)$
- Example:

$$\begin{aligned}
\gcd(1989, 867) &= \gcd(1989 - 2 \times 867, 867) \\
&= \gcd(255, 867) \\
&= \gcd(255, 867 - 3 \times 255) \\
&= \gcd(255, 102) \\
&= \gcd(255 - 2 \times 102, 102) \\
&= \gcd(51, 102) \\
&= \gcd(51, 102 - 2 \times 51) \\
&= \gcd(51, 0) \\
&= 51
\end{aligned}$$

Implementation

```
int gcd(int a, int b) {
    while(b){int r = a % b; a = b; b = r;}
    return a;
}
```

- Running time: $O(\log(a+b))$
- Be careful: `a % b` follows the sign of `a`
  - `5 % 3 == 2`
  - `-5 % 3 == -2`

Congruence & Modulo Operation

- $x \equiv y \pmod{n}$ means x and y have the same remainder when divided by n
- Multiplicative inverse
  - x^{-1} is the inverse of x modulo n if $xx^{-1} \equiv 1 \pmod{n}$
  - $5^{-1} \equiv 3 \pmod{7}$ because $5 \cdot 3 \equiv 15 \equiv 1 \pmod{7}$
  - May not exist (*e.g.*, inverse of 2 mod 4)
  - Exists if and only if $\gcd(x, n) = 1$

Multiplicative Inverse

- All intermediate numbers computed by Euclidean algorithm are integer combinations of a and b
  - Therefore, $\gcd(a, b) = ax + by$ for some integers x, y
  - If $\gcd(a, n) = 1$, then $ax + ny = 1$ for some x, y
  - Taking modulo n gives $ax \equiv 1 \pmod{n}$
- We will be done if we can find such x and y

Extended Euclidean Algorithm

- Main idea: keep the original algorithm, but write all intermediate numbers as integer combinations of a and b
- Exercise: implementation!

Chinese Remainder Theorem

- Given a, b, m, n with $\gcd(m, n) = 1$
- Find x with $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$
- Solution:
  - Let n^{-1} be the inverse of n modulo m
  - Let m^{-1} be the inverse of m modulo n
  - Set $x = ann^{-1} + bmm^{-1}$ (check this yourself)
- Extension: solving for more simultaneous equations

Outline

Binomial Coefficients

- $\binom{n}{k}$ is the number of ways to choose k objects out of n distinguishable objects
- same as the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$
  - Hence the name "binomial coefficients"
- Appears everywhere in combinatorics

Computing Binomial Coefficients

- Solution 1: Compute using the following formula:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

- Solution 2: Use Pascal's triangle
- Can use either if both n and k are small
- Use Solution 1 carefully if n is big, but k or $n-k$ is small

Fibonacci Sequence

- Definition:
  - $F_0 = 0$, $F_1 = 1$
  - $F_n = F_{n-1} + F_{n-2}$, where $n \geq 2$
- Appears in many different contexts

Closed Form

- $F_n = (1/\sqrt{5})(\varphi^n - \overline{\varphi}^n)$
  - $\varphi = (1 + \sqrt{5})/2$
  - $\overline{\varphi} = (1 - \sqrt{5})/2$
- Bad because φ and $\sqrt{5}$ are irrational
- Cannot compute the exact value of F_n for large n
- There is a more stable way to compute F_n
  - ... and any other recurrence of a similar form

Better "Closed" Form

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

- Use fast exponentiation to compute the matrix power
- Can be extended to support any linear recurrence with constant coefficients

Outline

Geometry

- In theory: not that hard
- In programming contests: more difficult than it looks
- Will cover basic stuff today
  - Computational geometry in week 9

When Solving Geometry Problems

- Precision, precision, precision!
  - If possible, don't use floating-point numbers
  - If you have to, always use `double` and never use `float`
  - Avoid division whenever possible
  - Introduce small constant ϵ in (in)equality tests
    - e.g., Instead of `if(x == 0)`, write `if(abs(x) < EPS)`
- No hacks!
  - In most cases, randomization, probabilistic methods, small perturbations won't help

2D Vector Operations

- Have a vector (x, y)
- Norm (distance from the origin): $\sqrt{x^2 + y^2}$
- Counterclockwise rotation by θ:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

  - Make sure to use correct units (degrees, radians)
- Normal vectors: $(y, -x)$ and $(-y, x)$
- Memorize all of them!

Line-Line Intersection

- Have two lines: $ax + by + c = 0$ and $dx + ey + f = 0$
- Write in matrix form:

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} c \\ f \end{bmatrix}$$

- Left-multiply by matrix inverse

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1} = \frac{1}{ae - bd} \begin{bmatrix} e & -b \\ -d & a \end{bmatrix}$$

  - Memorize this!
- Edge case: $ae = bd$
  - The lines coincide or are parallel

Circumcircle of a Triangle

- Have three points A, B, C
- Want to compute P that is equidistance from A, B, C
- Don't try to solve the system of quadratic equations!
- Instead, do the following:
  - Find the (equations of the) bisectors of AB and BC
  - Compute their intersection

Area of a Triangle

- Have three points A, B, C
- Want to compute the area S of triangle ABC
- Use cross product: $2S = |(B - A) \times (C - A)|$
- Cross product:

$$(x_1, y_1) \times (x_2, y_2) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

  - Very important in computational geometry. Memorize!

Area of a Simple Polygon

- Given vertices $P_1, P_2, \ldots, P_n$ of polygon P
- Want to compute the area S of P
- If P is convex, we can decompose P into triangles:

$$2S = \left| \sum_{i=2}^{n-1} (P_{i+1} - P_1) \times (P_i - P_1) \right|$$

- It turns out that the formula above works for non-convex polygons too
  - Area is the absolute value of the sum of “signed area”
- Alternative formula (with $x_{n+1} = x_1, y_{n+1} = y_1$):

$$2S = \left| \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i) \right|$$

Conclusion

- No need to look for one-line closed form solutions
- Knowing "how to compute" (algorithms) is good enough
- Have fun with the exercise problems
  - ... and come to the practice contest if you can!