String Algorithms

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Outline

String Matching Problem

- Given a text T and a pattern P, find all occurrences of P within T
- Notations:
  - n and m: lengths of P and T
  - Σ: set of alphabets (of constant size)
  - P_i: ith letter of P (1-indexed)
  - a, b, c: single letters in Σ
  - x, y, z: strings

Example

- $T =$ AGCATGCTGCAGTCATGCTTAGGCTA
- $P =$ GCT
- P appears three times in T

- A naive method takes $O(mn)$ time
  - Initiate string comparison at every starting point
  - Each comparison takes $O(m)$ time

- We can do much better!

Outline

String Matching Problem

Hash Table

Knuth-Morris-Pratt (KMP) Algorithm

Suffix Trie

Suffix Array

Hash Function

- A function that takes a string and outputs a number
- A good hash function has few collisions
  - *i.e.*, If $x \neq y$, $H(x) \neq H(y)$ with high probability
- An easy and powerful hash function is a polynomial mod some prime p
  - Consider each letter as a number (ASCII value is fine)
  - $H(x_1 \dots x_k) = x_1 a^{k-1} + x_2 a^{k-2} + \cdots + x_{k-1} a + x_k \pmod p$
  - How do we find $H(x_2 \dots x_{k+1})$ from $H(x_1 \dots x_k)$?

Hash Table

- Main idea: preprocess T to speedup queries
  - Hash every substring of length k
  - k is a small constant

- For each query P, hash the first k letters of P to retrieve all the occurrences of it within T

- Don't forget to check collisions!

Hash Table

- Pros:
  - Easy to implement
  - Significant speedup in practice
- Cons:
  - Doesn't help the asymptotic efficiency
    - Can still take $\Theta(nm)$ time if hashing is terrible or data is difficult
  - A lot of memory consumption

Outline

Knuth-Morris-Pratt (KMP) Matcher

- A linear time (!) algorithm that solves the string matching problem by preprocessing P in $\Theta(m)$ time
  - Main idea is to skip some comparisons by using the previous comparison result
- Uses an auxiliary array π that is defined as the following:
  - $\pi[i]$ is the largest integer smaller than i such that $P_1 \dots P_{\pi[i]}$ is a suffix of $P_1 \dots P_i$
- ... It's better to see an example than the definition

π Table Example (from CLRS)

i	1	2	3	4	5	6	7	8	9	10
P_i	a	b	a	b	a	b	a	b	c	a
$\pi[i]$	0	0	1	2	3	4	5	6	0	1

- $\pi[i]$ is the largest integer smaller than i such that $P_1 \dots P_{\pi[i]}$ is a suffix of $P_1 \dots P_i$
  - *e.g.*, $\pi[6] = 4$ since `abab` is a suffix of `ababab`
  - *e.g.*, $\pi[9] = 0$ since no prefix of length ≤ 8 ends with c
- Let's see why this is useful

Using the π Table

- $T =$ `ABC ABCDAB ABCDABCDABDE`
- $P =$ `ABCDABD`
- $\pi = (0, 0, 0, 0, 1, 2, 0)$
- Start matching at the first position of T:

```
12345678901234567890123
ABC ABCDAB ABCDABCDABDE
ABCDABD
1234567
```

- Mismatch at the 4th letter of P!

Using the π Table

- We matched $k = 3$ letters so far, and $\pi[k] = 0$
  - Thus, there is no point in starting the comparison at T_2, T_3 (crucial observation)
- Shift P by $k - \pi[k] = 3$ letters

```
12345678901234567890123
ABC ABCDAB ABCDABCDABDE
   ABCDABD
   1234567
```

- Mismatch at T_4 again!

Using the π Table

- We matched $k = 0$ letters so far
- Shift P by $k - \pi[k] = 1$ letter (we define $\pi[0] = -1$)

```
12345678901234567890123
ABC ABCDAB ABCDABCDABDE
    ABCDABD
    1234567
```

- Mismatch at T_{11}!

Using the π Table

- $\pi[6] = 2$ means $P_1 P_2$ is a suffix of $P_1 \dots P_6$
- Shift P by $6 - \pi[6] = 4$ letters

```
12345678901234567890123
ABC ABCDAB ABCDABCDABDE
    ABCDABD
        ||
        ABCDABD
        1234567
```

- Again, no point in shifting P by 1, 2, or 3 letters

Using the π Table

- Mismatch at T_{11} again!

```
12345678901234567890123
ABC ABCDAB ABCDABCDABDE
          ABCDABD
          1234567
```

- Currently 2 letters are matched
- Shift P by $2 - \pi[2] = 2$ letters

Using the π Table

- Mismatch at T_{11} yet again!

```
12345678901234567890123
ABC ABCDAB ABCDABCDABDE
          ABCDABD
          1234567
```

- Currently no letters are matched
- Shift P by $0 - \pi[0] = 1$ letter

Using the π Table

- Mismatch at T_{18}

```
12345678901234567890123
ABC ABCDAB ABCDABCDABDE
           ABCDABD
           1234567
```

- Currently 6 letters are matched
- Shift P by $6 - \pi[6] = 4$ letters

Using the π Table

- Finally, there it is!

```
12345678901234567890123
ABC ABCDAB ABCDABCDABDE
               ABCDABD
               1234567
```

- Currently all 7 letters are matched
- After recording this match (at $T_{16} \ldots T_{22}$, we shift P again in order to find other matches
  - Shift by $7 - \pi[7] = 7$ letters

Computing π

- Observation 1: if $P_1 \dots P_{\pi[i]}$ is a suffix of $P_1 \dots P_i$, then $P_1 \dots P_{\pi[i]-1}$ is a suffix of $P_1 \dots P_{i-1}$
  - Well, obviously...
- Observation 2: all the prefixes of P that are a suffix of $P_1 \dots P_i$ can be obtained by recursively applying π to i
  - *e.g.*, $P_1 \dots P_{\pi[i]}$, $P_1 \dots, P_{\pi[\pi[i]]}$, $P_1 \dots, P_{\pi[\pi[\pi[i]]]}$ are all suffixes of $P_1 \dots P_i$

Computing π

- A non-obvious conclusion:
  - First, let's write $\pi^{(k)}[i]$ as $\pi[\cdot]$ applied k times to i
  - *e.g.*, $\pi^{(2)}[i] = \pi[\pi[i]]$
  - $\pi[i]$ is equal to $\pi^{(k)}[i-1]+1$, where k is the smallest integer that satisfies $P_{\pi^{(k)}[i-1]+1} = P_i$
    - If there is no such k, $\pi[i] = 0$
- Intuition: we look at all the prefixes of P that are suffixes of $P_1 \dots P_{i-1}$, and find the longest one whose next letter matches P_i

Implementation

```
pi[0] = -1;
int k = -1;
for(int i = 1; i <= m; i++) {
  while(k >= 0 && P[k+1] != P[i])
    k = pi[k];
  pi[i] = ++k;
}
```

Pattern Matching Implementation

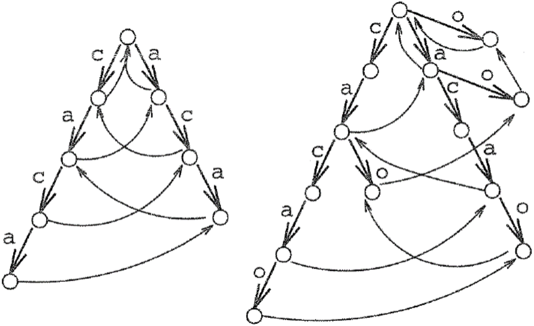
```
int k = 0;
for(int i = 1; i <= n; i++) {
  while(k >= 0 && P[k+1] != T[i])
    k = pi[k];
  k++;
  if(k == m) {
    // P matches T[i-m+1..i]
    k = pi[k];
  }
}
```

Outline

Suffix Trie

- Suffix trie of a string T is a rooted tree that stores all the suffixes (thus all the substrings)
- Each node corresponds to some substring of T
- Each edge is associated with an alphabet
- For each node that corresponds to ax, there is a special pointer called *suffix link* that leads to the node corresponding to x
- Surprisingly easy to implement!

Example



(Figure modified from Ukkonen's original paper)

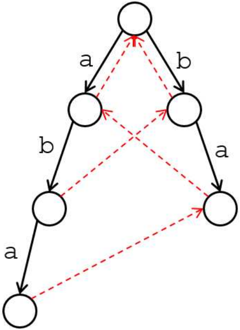
Incremental Construction

- Given the suffix tree for $T_1 \dots T_n$
  - Then we append $T_{n+1} = a$ to T, creating necessary nodes
- Start at node u corresponding to $T_1 \dots T_n$
  - Create an a-transition to a new node v
- Take the suffix link at u to go to u', corresponding to $T_2 \dots T_n$
  - Create an a-transition to a new node v'
  - Create a suffix link from v to v'

Incremental Construction

- Repeat the previous process:
  - Take the suffix link at the current node
  - Make a new a-transition there
  - Create the suffix link from the previous node
- Stop if the node already has an a-transition
  - Because from this point, all nodes that are reachable via suffix links already have an a-transition

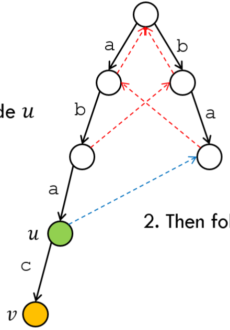
Construction Example



Given the suffix trie for `aba`
We want to add a new letter `c`

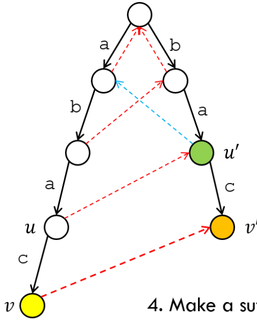
Construction Example

1. Start at the green node u and make a $\mathtt{c}$-transition



2. Then follow the suffix link

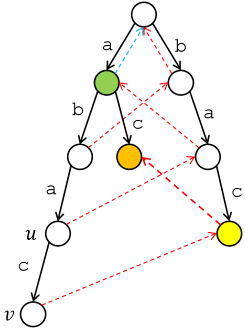
Construction Example



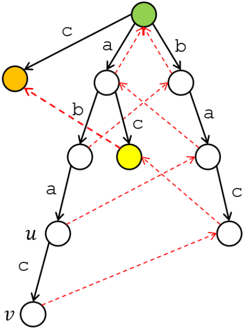
3. Make a c-transition at u'

4. Make a suffix link from v

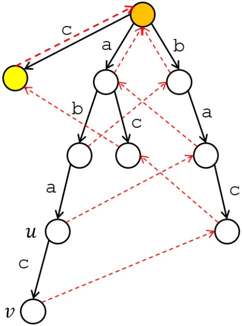
Construction Example



Construction Example



Construction Example



Construction Example

- Construction time is linear in the tree size
- But the tree size can be quadratic in n
  - *e.g.*, $T = \texttt{aa}\ldots\texttt{abb}\ldots\texttt{b}$

Construction Example

- To find P, start at the root and keep following edges labeled with P_1, P_2, etc.
- Got stuck? Then P doesn't exist in T

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Suffix Trie

Suffix Array

Suffix Array

Input string	Get all suffixes	Sort the suffixes	Take the indices
BANANA	1 BANANA 2 ANANA 3 NANA 4 ANA 5 NA 6 A	6 A 4 ANA 2 ANANA 1 BANANA 5 NA 3 NANA	6,4,2,1,5,3

Suffix Array

- Memory usage is $O(n)$
- Has the same computational power as suffix trie
- Can be constructed in $O(n)$ time (!)
  - But it's hard to implement
- There is an approachable $O(n \log^2 n)$ algorithm
  - If you want to see how it works, read the paper on the course website
  - `http://cs97si.stanford.edu/suffix-array.pdf`

Notes on String Problems

- Always be aware of the null-terminators
- Simple hash works so well in many problems
- If a problem involves rotations of some string, consider concatenating it with itself and see if it helps
- Stanford team notebook has implementations of suffix arrays and the KMP matcher