A Sufficient Statistics Approach for Aggregating Firm-Level Experiments *

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Abstract

We consider a dynamic economy populated by heterogeneous firms subject to generic capital frictions: adjustment costs, taxes and financing constraints. A random subset of firms in this economy receives an empirical "treatment", which modifies the parameters governing these frictions. An econometrician observes the firm-level response to this treatment, and wishes to calculate how macroeconomic outcomes would change if *all* firms in the economy were treated. Our paper proposes a simple methodology to estimate this aggregate counterfactual using firm-level evidence only. Our approach takes general equilibrium effects into account, requires *neither* a structural estimation *nor* a precise knowledge on the exact nature of the experiment and can be implemented using simple moments of the distribution of revenue-to-capital ratios. We provide a set of sufficient conditions under which these formulas are valid and investigate the robustness of our approach to multiple variations in the aggregation framework.

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1 Introduction

Governments around the world have a wide range of policies to facilitate business investment and growth. A burgeoning empirical literature seeks to evaluate the effectiveness of these policies using firm-level data and well-identified empirical settings. Some papers look at financial reforms (see for instance Aghion et al. (2007), Bertrand et al. (2007), Ponticelli and Alencar (2016), Larrain and Stumpner (2017)). Others analyze firm response to the availability of subsidized credit (e.g. Lelarge et al. (2010), Banerjee and Duflo (2014), Brown and Earle (2017)), or changes in bank lending behavior (Fraisse et al. (2017), Blattner et al. (2017)). Another set of papers study the effect of capital taxes or subsidies on firm investment and hiring (Yagan (2015), Zwick and Mahon (2015), Giroud and Rauh (2016), Rotemberg (2017)). By comparing treated firms to a plausibly exogenous control group, these papers quantify the relative effect of these policy interventions on *treated* firms. However, they remain silent on how these firm-level effects would aggregate, were the intervention generalized to a broader set of firms. In this paper, we offer simple formulas to estimate such an aggregate counterfactual using firm-level evidence. This approach does not require the estimation of a structural model of firm behavior and, in particular, does not require that the empiricist precisely knows how the intervention affects firm-level distortions.

Aggregating firm-level responses to a scaled-up version of these policies is a non-trivial exercise. First, scaling up a policy may change the microeconomic effect of this policy as it may interact with other distortions (for instance, subsidizing credit may interact with corporate taxes) that are themselves affected by general equilibrium effects. Second, standard equilibrium effects will typically dampen firm-level responses: for instance, if a growth-enhancing policy is extended to a larger set of firms, labor demand increases, which in turn raises the equilibrium wage and mitigates the initial direct effect. Third, extending the policy to a larger scale reallocates inputs across firms: as distortions are reduced, capital and labor flow from firms with low marginal productivity to firms with high marginal productivity, which leads to an increase in aggregate productivity. This paper provides simple formulas that allow to take these effects into account, without having to estimate a structural model of firm behavior.

To derive these formulas, we proceed in three steps. First, we set up a general equilibrium model with heterogeneous firms who face stochastic productivity shocks and are subject to several forms of distortions: adjustment costs, taxes and financing frictions. We relate aggregate output and total factor productivity (TFP) to the economy-wide distribution of revenue to capital ratios. Under some assumptions, the distribution of the revenue to capital ratio[1] captures the extent of distortions in the economy: A firm with a relatively high revenue to capital ratio is a firm that invests too little, because of adjustment costs, financing constraints or taxes. The distribution of revenue to capital ratios is not a deep structural parameter, in the sense that it depends on firms' histories and choices. But the effect of a given policy on this distribution can easily be estimated using standard datasets and a well-designed experimental setting.

In a second step, we assume that a small-scale policy intervention targets a random subset of firms in the economy. This treatment affects parameters governing firm-level frictions, but we do not need to specify which ones exactly. Using firm-level data, we assume that an econometrician estimates the effect of such a treatment on the empirical distribution of revenue to capital ratios. The problem is, however, that such treatment effects are measured in *partial equilibrium*.

A key contribution in this paper is to provide conditions under which these treatment effects on revenue to capital ratios are *independent* of general equilibrium conditions. This property allows us to safely inject the policy effect estimated in partial equilibrium into the formulas for GE outcome. It relies on two key assumptions. First, the sources of distortions (financing frictions and constraints, tax schedules, real and financial adjustment costs) are assumed to be homogeneous of degree 1. The intuition for this is that homogeneity guarantees that frictions remain on average constant on a size-adjusted basis. Hence, a change in general equilibrium, which affects firm size, will not affect distortions. The second assumption is the firm-level Cobb-Douglas production function, with either constant or decreasing returns to scale. As we show in our literature review in Table 1, these assumptions are consistent with 80-90% of the models of firm dynamics employed in structural corporate finance and macroeconomics.

The formulas we obtain for aggregate output and TFP combine parameters of the macroeconomic model (labor share, goods substitutability, labor supply elasticity) and three sufficient statistics for the joint distribution of log productivity and log revenue to capital. The first statistic is the effect of the treatment on average log wedge. It captures the extent to which the treatment affects the aggregate amount of savings available to firms. The second statistic is the treatment effect on the variance of log wedges. It measures how the treatment distorts the allocation of capital across firms. The final statistic is the treatment

[1]The revenue to capital ratio is commonly called "marginal revenue product of capital" (MRPK) or equivalently "capital wedge" in the misallocation literature (Restuccia and Rogerson (2008), Hsieh and Klenow (2009)).

effect on the covariance of log wedges and log productivities. Intuitively, if the treatment reduces this covariance, it will make the productive firms relatively less distorted which is good for aggregate output.

We then consider a series of relevant extensions to the basic setup and show how our aggregation formulas extend to (most of) these different settings. We show that our key invariance property is robust to (1) the presence of unobserved productivity heterogeneity, (2) the presence of heterogeneous treatment effects and (3) various extensions of our baseline industry structure (heterogeneous industries, roundabout production structure with an input-output matrix, heterogeneous markups). In these extensions, our baseline formulas can be more complex but remain highly tractable and easy to use. We also investigate, in Appendix, how labor distortions affect our methodology. We show that the invariance property continues to holds as long as labor distortions are generated by exogeneous taxes (e.g. payroll taxes) or variations in minimum wage. In this case, our formula include additional sufficient statistics that account for the interaction between capital and labor distortions. Using our framework, we also explore the aggregation of treatments designed to increase firm productivity instead of targeting distortions. Since productivity changes may interact in a complex way with existing distortions, aggregation is non-trivial. We leverage the invariance property to compute simple aggregation formulas. Besides the effect of the policy on productivity per se, these formula require estimation of how productivity changes interact with and affect distortions.

Finally, we extend our formula to the case where production functions are CES. In this case, our invariance property fails to hold because the multiplicative property of the Cobb-Douglas model is lost. But there is a way around this that this analysis clarifies. Aggregation remains feasible if one computes two additional sufficient statistics which in esssence capture the feedback effect of the aggregate equilibrium on the mean treatment effect. We provide a tractable aggregation formula that include these statistics as well as as measured of direct treatment effect on wedges. In the CES framework, wedges need to be computed differently that in the Cobb-Douglas framework. We clarify this too.

Our paper is first and foremost of interest for the growing literature that empirically analyzes firm-level distortions using experimental-like settings. Many of the papers cited above estimate the *firm-level* effect of policies promoting business investment but do not speak to how these policies would affect macroeconomic outcome were the policy extended to all firms in the economy. Our paper provides a simple framework to answer this question using similar identification strategies but focusing on a set of sufficient statistics typically not computed in these studies (mean log revenue to capital ratio, its variance, and its covariance with log productivity). Recent exceptions are Blattner et al. (2017), Rotemberg (2017) and Larrain and Stumpner (2017), who consider an aggregation framework somewhat similar to ours, but in which the distribution of revenue to capital ratios are assumed to be exogenous, and in particular independent of aggregate conditions. One of our contributions is to provide sufficient conditions under which endogenous capital wedges are in fact independent of the market equilibrium. Our paper is also related to the rising literature that seeks to bridge reduced form analysis and structural approach by isolating simple "sufficient statistics" that help measure aggregate outcomes out of simple firm or household-level statistics. For instance, Davila (2016) writes down a model of household borrowing with costs of bankruptcy. He derives the optimal bankruptcy exemption as a function of sufficient statistics that can be observed, in particular, the reduction of consumption measured among bankrupt households and the probability of defaulting. More closely related is recent work by Baqaee and Farhi (2017), who derive sufficient statistics to aggregate micro-level estimates. Their approche diverges from ours in that they use a very general aggregation framework, but, in the spirit of Restuccia and Rogerson (2008) and Hsieh and Klenow (2009) assume distortions can be represented by exogenous wedges over the cost of capital. With respect to these papers, our contribution is twofold. First, we take on board the fact that capital wedges are most of the time *not* exogenous but result from complex firm decision. Yet, we are able to prove that under a large class of models used in firm dynamics and macro finance that these wedges are invariant to "scaling up" the experiment. Thus, the micro-empiricist can use them to recover the aggregate effect of a given intervention. In doing so, the empiricist is able to retrieve the aggregate impact of a change in some frictions separately from real frictions (adjustment costs). This is our second contribution to this literature.

Section 2 lays out the economic model. Section 3 develops our methodology. Section 4 shows that the assumptions of Section 2, which are necessary to this result, are consistent with most of the literature on firm dynamics. Section 5 investigates the robustness of our formulas to various extensions to the basic set-up. The last Section concludes.

2 The Economic Model

2.1 Set-up

The economy is dynamic ($t = 0,\ 1, \ldots, \infty$), but there is no aggregate uncertainty and the economy is assumed to be in steady state. We first consider a simple market structure and extend the analysis to include heterogeneous industries in Section 5. At each date t, a continuum of monopolists, indexed by $i \in [0;1]$, produce imperfectly substitutable intermediate goods in quantity y_{it} at a price p_{it} (Dixit and Stiglitz (1977)). There is a perfectly competitive final good market, which aggregates intermediate output according to a CES technology:

$$Y = \left(\int_i y_{it}^{\theta} di \right)^{\frac{1}{\theta}}, \tag{1}$$

where we omit the t subscript for aggregate output because the economy is in steady state. We use the final good as the numeraire. Profit maximization in the final good market implies that the demand for product i is given by: $p_{it} = \left(\frac{Y}{y_{it}}\right)^{1-\theta}$ and $-\frac{1}{1-\theta}$ is the price elasticity of demand.

To produce, firms combine labor and capital according to a Cobb-Douglas production function: $y_{it} = e^{z_{it}} k_{it}^{\alpha} l_{it}^{1-\alpha}$, where k_{it} is firm i's capital stock in period t, l_{it} is the labor input in period t, α is the capital share. Log productivity $z_{it} \in Z$ is independent *across* firms, but follows *within* firm a Markovian process that is common to all firms. In Section 5.1, we show that the results proven in this Section still hold if firms have heterogeneous long-term productivity fixed effects.

With monopolistic competition and the demand system in Equation (1), firm i revenue in period t is $p_{it} y_{it} = Y^{1-\theta} y_{it}^{\theta}$. We assume that there is no adjustment cost to labor so that labor is a static input. If w is the steady state wage, static labor optimization implies that firm i's profit becomes:

$$\pi(z_{it}, k_{it}; w, Y) = \max_{l} \left(Y^{1-\theta} e^{z_{it}} k_{it}^{\theta\alpha} l^{\theta(1-\alpha)} - wl \right) = \Omega \left(\frac{Y^{1-\phi}}{w^{\phi \frac{1-\alpha}{\alpha}}} \right) e^{\frac{\phi}{\alpha} z_{it}} k_{it}^{\phi} ,$$

where $\phi = \frac{\alpha\theta}{1-(1-\alpha)\theta} < 1$, and Ω is a constant.

We call Θ the vector of deep structural parameters describing the firm's (real and financial) frictions on capital. In this Section, we assume to clarify intuitions that all firms have the same parameter, but relax this assumption in Section 5, where we explore the consequence of firm heterogeneity on our aggregation formulas.

The capital good is the final good – so that its price is also 1 – but using it in production leads to physical depreciation at rate δ. Capital investment in period t is subject to a one period time-to-build. Firms can finance investment using the profits they realize from operations or through external financing. The first source of outside financing is one-period debt. b_{it+1} is the total real payment due to creditors in period $t+1$. To simplify notations, we define $\boldsymbol{x}_{it} = (k_{it}, k_{it+1}, b_{it}, b_{it+1})$. We note r_{it} the interest rate charged by lenders (more below), so that $\frac{b_{it+1}}{1+r_{it}}$ is the proceed from debt financing received in period t. We allow the firm's investment and debt financing at date t to be subject to adjustment costs $\Gamma\left(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w, Y\right)$. We also assume that firms pay taxes and receive subsidies: $\mathcal{T}(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w, Y)$ corresponds to the net tax paid by the firm.

Finally, we allow for generic forms of financing frictions. First, equity issuance may be costly, and we note such costs $\mathcal{C}(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w, Y)$. These costs are obviously zero when the firm does not issue equity, which in these models happens as soon as cash-flows are positive. Second, the amount of outside financing may be constrained, a friction that we capture through a vector of constraint: $\boldsymbol{M}(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w, Y) \leq 0$. Third, the interest rate on debt is described the function $r()$ such that r_{it}=$r(z_{it}, \boldsymbol{x_{it}}; \boldsymbol{\Theta}, w, Y)$. This function may allows for risky debt. It may also embed costs of financial distress such as liquidation costs for instance, and thus reflect financing frictions.

Given all the above, we note e_{it} the cash-flows of the firm, net of equity issuance costs:

$$
\begin{aligned}
e_{it} &= \pi(z_{it}, k_{it}; w, Y) - (k_{it+1} - (1-\delta)k_{it}) - \Gamma\left(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w, Y\right) \\
&+ \left(\frac{b_{it+1}}{1 + r(z_{it}, \boldsymbol{x_{it}}; \boldsymbol{\Theta}, w, Y)} - b_{it}\right) - \mathcal{T}(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w, Y) \\
&- \mathcal{C}(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w, Y) \\
&= e(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w, Y)
\end{aligned}
$$

The timing is standard in models of firm dynamics. At the beginning of period t, productivity z_{it} is realized. The firm then combines capital in place k_{it} with labor l_{it} to produce and receive the corresponding profits. It then selects the next period stock of capital k_{it+1}, pays the corresponding adjustment costs, reimburses its existing debt b_{it} and receives the proceeds from debt issuance $\frac{b_{it+1}}{1+r_{it}}$. Then, the period ends. We allow for default, priced in the interest rate of debt (as discussed above): To model this, we assume that the firm survives into the following period if and only if productivity belongs to a "survival set":

$z_{it+1} \in \mathcal{Z}(k_{it+1}, b_{it+1}; \mathbf{\Theta}, w, Y)$. For z_{it+1} outside of the survival set, we assume to simplify notations that default occurs and continuation value to the firm's owner is zero.[2]

To lighten the notation, let us temporarily omit now the it index and denote with prime next-period variables. This dynamic problem has one exogenous state variable z and two endogeneous state variables k and b. To ensure that policy functions are defined on a compact set, we make the following assumptions:

Assumption 1 (Technical assumptions)**.** *Let Y be the aggregate demand shifter and w the wage. Then, state variables (z, k, b) satistfy the following assumptions:*

1. *the cash-flow function $e(.; \mathbf{\Theta}, w, Y)$ is a piecewise continuous function of (z, k, k', b, b')*

2. *log productivity $z \in Z$ where Z is compact and convex. The Markovian transition function underlying z is strictly positive, has no atom and satisfies the Feller property.*

3. *capital takes values in the set K which is assumed to be convex and compact.*

4. *debt values b are restricted to a compact and convex set B.*

5. *Conditionnally on past (k, b) and current capital choice k', debt values b' are restricted to a correspondence set $\mathcal{B}$:*

$$\mathcal{B}(z, k, k', b; \mathbf{\Theta}, w, Y) = B \cap \{b' | \boldsymbol{M}(z, k, k', b, b'; \mathbf{\Theta}, w, Y) \leq 0\}$$

We assume that the financial constraint $\boldsymbol{M}$ is such that $\mathcal{B}$ is compact, convex and non-empty.

To ease notations, we write $\mathcal{B}(z, k, k', b; \Theta, w, Y) \equiv \mathcal{B}(z, k, k', b)$.

The first item only assumes piecewise continuity because models may include (1) fixed costs of equity issuance (e.g. Hennessy and Whited (2007)) or (2) fixed capital adjustment costs (e.g. Caballero and Leahy (1996)). Most papers typically prove the third item of the above assumption by picking a maximum capital stock that the firm will never find optimal to choose. Decreasing returns to scale ensure that all levels of capital above this maximum cannot be optimal either (e.g. Hennessy and Whited (2007)). The fourth item is typically obtained in corporate finance models through assuming that the return on cash

[2]Our results would carry through for a larger class of default continuation values, as long as they satisfy an homogeneity property similar to the assumptions of proposition 2. We do not explictly write down this model to lighten notations.

is low compared to the shareholder's discount rate, which bounds debt b from below. b is bounded above through financing frictions (cost of financial distress, collateral or cash-flow constraints).

Define $\mathcal{C}(\Theta, w, Y)$ the set of continuous and bounded functions $f(z, k, b; \Theta, w, Y) : Z \times K \times B \to \mathbb{R}$. Following the rest of the literature, we define the Bellman operator T on the functional space $\mathcal{C}(\Theta, w, Y)$:

$$Tf(z,k,b;\boldsymbol{\Theta},w,Y) = \left| \begin{array}{l} \max\limits_{k',b' \in K \times B} e(z,k,k',b,b';\boldsymbol{\Theta},w,Y) + \beta \mathbb{E}_{z' \in \mathcal{Z}(k',b';\boldsymbol{\Theta},w,Y)}[f(z',k',b';\boldsymbol{\Theta},w,Y)|z], \\ \boldsymbol{M}(z,k,k',b,b';\boldsymbol{\Theta},w,Y) \leq 0 \end{array} \right. \tag{2}$$

where $\beta < 1$ is the firm's discount rate.

The firm is assumed to maximize the expected present value of equity cash-flows, so that the equity value function is the fixed point of the Bellman operator T. The maximands are the optimal debt and capital policy functions. Under assumption 1, the contraction mapping theorem holds and the operator T has only one fixed point in the space of continuous and bounded functions $\mathcal{C}(\Theta, w, Y)$ which is the market value of equity. This unique solution defines unique policy functions so that the problem is well-defined. [3]

The household side of the economy is stripped down to its essentials. A representative household has GHH preferences (Greenwood et al. (1988)) over consumption and leisure: $u(c_t, l_t) = \frac{1}{1-\gamma}\left(c_t - \frac{l_t^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \frac{\bar{w}}{\bar{L}^{\frac{1}{\epsilon}}}\right)^{1-\gamma}$, where c_t is period t consumption, l_t is period t labor supply, ϵ is the Frisch elasticity and $(\bar{w}, \bar{L})$ are normalizing constants. The representative household owns all the firms in the economy, as well as a safe asset that offers real return r. β^H is the representative household's discount rate. In the absence of aggregate uncertainty, optimal consumption and labor supply decisions imply that $L_t^s = \bar{L}\left(\frac{w}{\bar{w}}\right)^{\epsilon}$ and $\beta^H = \frac{1}{1+r}$, where r is the risk-free rate. Note that since households portfolios are well diversified across firms, we also have $\beta = \frac{1}{1+r}$ even though the model potentially allows for firms' default.

[3]These results are broadly used in the literature. They arise from the extension of the results in Stokey and Lucas (1989) from strictly continuous to piecewise continuous cash-flow functions. See Caballero and Leahy (1996) for a version of proof with fixed adjustment costs and Hennessy and Whited (2007) for a version of the proof with fixed equity issuance costs. In both cases, the logic is the same and applies whenever the cash-flow is piecewise continuous.

2.2 Introducing Capital Wedges

Instead of solving the model explicitly, we will characterize its equilibrium as a function of the distribution of objects defined as capital wedges τ which vary over time and across firms. These wedges are defined as the ratio of a firm's marginal revenue product of capital to some frictionless user cost of capital R for firm i in period t. R is arbitrary but fixed throughout the analysis.

Definition 1 (Definition of capital wedges)**.** *For each firm-year observation, we define the wedge τ_{it} as:*

$$1 + \tau_{it} = \frac{\alpha\theta}{R}\frac{p_{it}y_{it}}{k_{it}}$$

We note that capital wedges are *not* welfare-based notion but pure empirical constructs. The capital wedge essentially captures how much the capital stock of a firm deviates from frictionless optimization. It is equal to zero when the firm invests as if there were no frictions at all. In our model, firms potentially deviate from their frictionless optimum for three reasons: financing frictions, adjustment costs and taxes. It is important to note that these wedges do not generically have a direct welfare interpretation. As recently noted by Asker et al. (2014), adjustment costs or time to build in capital can drive a wedge between marginal productivity and cost of capital, even if the economy if at the first best optimum. This is because adjustment costs are real frictions. Financing frictions of various sorts also do not generically have a welfare interpretation. For instance, financing constraints may arise in order to overcome information asymmetries in financing. So while financing constraint related wedges indicate that the economy is not at the first best optimum, the economy may still be operating at the constrained optimum. The same could be said of taxes, as tools to raise taxes in a non-distortionary way may fail to exist due to information asymmetries for example.

While not welfare-related, wedges are easy to measure since they are proportional to the ratio of revenue to capital, and, as we will see below, they are relevant for aggregation. Both revenue and capital can be measured in standard firm-level datasets containing financial statements. Of course, wedges are *not* exogenous: They are technically, at the firm level, a function of the state variables (k and b) as well as the aggregate state of the economy (summarized by w and Y) and the parameter vector Θ. In what follows, we take inspiration from existing literature and offer formulas that connect aggregate output and TFP with the distribution of wedges.

2.3 Equilibrium

We focus on two aggregate measures relevant to the literature: Real output Y, and aggregate Total Factor Productivity defined as $\frac{Y}{K^{\alpha}L^{1-\alpha}}$, where K and L are the aggregate capital and labor stocks. The TFP definition is the same as the one used in the recent literature on misallocation among heterogeneous firms (e.g. Restuccia and Rogerson (2008), Hsieh and Klenow (2009), Moll (2014)).[4]

While the measures of productive efficiency we focus on are standard, an important drawback is that they are quite different from welfare, and this for several different reasons. First, welfare in our model is given by the utility of the representative agent and is proportional to $\left(C - \frac{L^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \frac{\bar{w}}{\bar{L}^{\frac{1}{\epsilon}}}\right)^{1-\gamma}$ where C is steady state consumption and L is aggregate employment. Welfare thus differs from output along several dimensions: it does not account for disutility of labor, for investment or for friction-related costs. It also differs from TFP in the sense that it does not penalize the use of capital and labor in the same way. Welfare formulas are actually very similar to the formulas we will derive for TFP and output, and exploit the same sufficient statistics. We do not supply then in the paper to keep exposition crisp and because we believe the model is better designed to describe the productive structure of the economy. They are, however, available upon request.

In this set-up, it is possible to write aggregate output and total factor productivity as functions of the distribution of wedges and log productivities. In order to do this, we simply follow the existing literature (e.g. Hsieh and Klenow (2009), Midrigan and Xu (2014)) and write down the market clearing conditions. Simple algebra (detailed in Appendix A.1) leads to:

[4] This TFP measure is identical to the one defined in Baqaee and Farhi (2017), whereby factor weights are given distortion-weighted cost shares. Aggregate distortion-weighted capital cost is given by:

$$\widetilde{RK} = \int R(1+\tau_i)k_i di = \int \theta\alpha p_i y_i di = \theta\alpha Y$$

Labor faces no distortion in this baseline so that aggregate labor costs are given by: $wL = \int wl_i di = \theta(1-\alpha)Y$. Hence, the distortion-weighted capital share is indeed α, while the distortion-weighted labor share is just $1-\alpha$.

$$Y \propto \left(\int_i \left(\frac{e^{z_{it}}}{(1+\tau_{it})^{\alpha}} \right)^{\frac{\theta}{1-\theta}} di \right)^{(1+\epsilon)\frac{1-\theta}{(1-\alpha)\theta}} \tag{3}$$

$$TFP = \frac{Y}{K^{\alpha}L^{1-\alpha}} \propto \frac{\left(\int_i \left(\frac{e^{z_{it}}}{(1+\tau_{it})^{\alpha}} \right)^{\frac{\theta}{1-\theta}} di \right)^{\frac{1-(1-\alpha)\theta}{\theta}}}{\left(\int_i \frac{e^{\frac{\theta}{1-\theta} z_{it}}}{(1+\tau_{it})^{\frac{1-(1-\alpha)\theta}{1-\theta}}} di \right)^{\alpha}} \tag{4}$$

where we omit subscripts t since in steady state aggregate output Y, aggregate capital stock K and aggregate employment L do not vary with time, and the joint distribution of log productivities z_{it} and wedges τ_{it} is also stationary. This is because there is a continuum of firms and productivity shocks are independent across firms.

Note that these equations do *not* solve the model. They only impose the constraints that: (1) firm-level production is Cobb-Douglas and is aggregated via the CES aggregator (1) and (2) product and labor markets clear. They do not make any other assumption about the frictions that firms are facing. In this framework, all the frictions are "summarized" by the capital wedges in Definition 1. The above equations simply describe a relationship between Y and TFP on the one hand, and the joint steady state distribution of z_{it} and τ_{it}. They do not *solve* for the equilibrium since this steady state distribution is itself a complicated function of the deep structural parameters Θ and the equilibrium outcome Y. Truly solving for this is in general not feasible analytically and requires numerical approximations. But for our purpose, the advantage of the above formulation is that the wedge distribution is *observable* in the data, so that, under assumptions detailed below, an empiricist can use them to connect changes in this distribution with changes in the above macro variables. But before describing this, we first introduce a simplification in order to make these formulas easier to apprehend.

2.4 Small Perturbation Approximation

In what follows, we focus on the following case:

Assumption 2. *We assume that* $z_{it} \ll 1$ *and* $\log(1+\tau_{it}) \ll 1$. *All analytical results in the paper rely on second order Taylor expansions of these quantities.*

In this multiplicative set-up, Assumption 2 is equivalent to assuming that $\log(1+\tau_{it})$ and z_{it} are jointly normally distributed (which is the assumption made for instance in

Hsieh and Klenow (2009)). Since $\log\left(\frac{p_{it}y_{it}}{k_{it}}\right) = \log(1+\tau_{it}) + \text{cst}$, Assumption 2 implies that the log revenue to capital ratio is also normally distributed. We test the relevance of this assumption using data from BvD AMADEUS Financials for the year 2014. As in Gopinath et al. (2015), we measure $p_i y_{it}$ as the value added of the firm, i.e. the difference between gross output (operating revenue) and materials. We measure the capital stock, k_{it}, with the book value of fixed tangible and intangible. For 9 countries in our sample (France, Italy, Spain, UK, Portugal, Croatia, Sweden, Bulgaria and Romania), we report in Figure 1 normal probability plots, i.e. plots of the empirical c.d.f. of the standardized log revenue to capital ratios against the c.d.f. of a normal distribution. Figure 1 shows that the log-normality assumption is a reasonable one.

Neither of these assumptions (small deviations or log normality) is necessary to our approach and we provide in Appendix B.3 formulas that do not rest on it. But assumption 2 proves useful to clarify the logic of our approach, as it summarizes the distribution of wedges in a handful of moments.

The second order Taylor expansion of equations (3-4) in $\log(1+\tau)_{it}$ and z_{it} around their means leads to simple formulas summarized in the following proposition:

Proposition 1. *Under Assumption 2, at equilibrium, aggregate output and TFP can be written as simple functions of three moments of the joint distribution of log wedges* $\ln(1+\tau_{it})$ *and log productivity* z_{it}*:*

$$\log Y = \frac{\alpha(1+\epsilon)}{1-\alpha}\left(-\mu_\tau(\Theta,w,Y) + \frac{1}{2}\frac{\theta}{1-\theta}\left(\alpha\sigma_\tau^2(\Theta,w,Y) - 2\sigma_{\tau z}(\Theta,w,Y)\right)\right) + cst \tag{5}$$

$$\log(TFP) = -\frac{\alpha}{2}\left(1+\frac{\alpha\theta}{1-\theta}\right)\sigma_\tau^2(\Theta,w,Y) \tag{6}$$

where $\mu_\tau(\Theta,w,Y)$ *and* $\sigma_\tau^2(\Theta,w,Y)$ *are the steady-state mean and variance of the steady-state distribution of log capital wedges.* $\sigma_{\tau z}(\Theta,w,Y)$ *is the covariance between log productivity and log wedges. All three moments are generically functions of deep structural parameters* Θ*, aggregate output* Y *and the market clearing wage* w*.*

Proof. See Appendix A.2. □

These formulas illustrate forces already discussed in the literature. Dispersion of wedges impairs aggregate efficiency because it creates capital misallocation (Hsieh and Klenow (2009)). A positive correlation between productivity and wedges also hurts aggregate production: output is lower when the most productive firms experience the largest distortions (Hopenhayn (2014)). However, in our setting, such a correlation does not affect aggregate TFP. This result emanates from the small deviation assumption (or alternatively, from log normality). For instance, it does not hold in Restuccia and Rogerson (2008), who use a binary distribution for the distribution of distortions.

The above formulas suggest a very simple methodology to aggregate firm-level evidence:

1. Measure the treatment effect of a policy experiment on the three moments introduced in Proposition 1 (mean and variance of log wedges, and covariance of log wedges with log productivity). These three moments are easy to compute using firm-level data since log wedges are equal to log revenue-to-capital ratios up to a constant.

2. Plug these treatment effects into formulas (5-6). This would lead to the aggregate effect (in terms of log-changes in aggregate output and TFP) of generalizing the experiment to all firms in the economy. This approach is originally the one of Hsieh and Klenow (2009) who use the variance of log revenue to capital ratios of the US to investigate TFP losses among Indian firms due to misallocation. It has since been taken on in a few recent papers based on quasi-experimental frameworks (Blattner et al. (2017), Larrain and Stumpner (2017), Rotemberg (2017)).

However, this methodology faces a challenge, well illustrated by the above formulas. The distribution of wedges is only observed *conditional* on the economy's equilibrium (w, Y). The inference on the joint distribution of $(z_{it}, \log(1+\tau_{it}))$ in step 1 may thus depend on the size of the experiment, as we expect aggregate outcomes to change as the experiment scales up. In particular, when all firms receive the treatment, the equilibrium distribution of $(z_{it}, \log(1+\tau_{it}))$ may differ from the distribution observed in an experimental setting where only a fraction of firms receive the treatment. The next section details this issue and shows a set of sufficient conditions for the model presented in this section such that this issue does not arise and the above methodology is valid.

3 Inference and Aggregation of Policy Experiments

In this section, we justify the use of firm-level estimates to perform aggregate counterfactuals. To simplify the exposition, we assume in this section that the empirical exercise consists of a simple binary treatment, where a random subset of firms are treated. Our approach can easily be generalized to continuous treatments and heterogeneous response (we explore these extensions in Section 5)

3.1 Definition of the Empirical Treatment

An econometrician observes data on an infinite number of firms, of which a subset is subject to an empirical treatment. The treatment is binary: firm i is either treated ($T_i = 1$) or untreated ($T_i = 0$). This treatment is a policy that affects the parameters Θ governing financing constraints, adjustment costs or taxes. Θ_0 (resp. Θ_1) correspond to the parameters of non-treated (resp. treated) firms. The econometrician does not necessarily know how the treatment affects these parameters. However, we assume that she knows that the treatment leaves the following three parameters unchanged: the capital share in production α, the price elasticity of demand θ, and the labor supply elasticity ϵ.

To be consistent with our model, we assume that the econometrician observes the ergodic distribution of firms in the economy. As a result of this assumption, we do not need to worry about the exogeneity of the treatment: at the steady state, the model converges to its new ergodic distribution, which is independent of initial conditions. This arises because, in our simple setting, heterogeneity only stems from temporary productivity shocks. Our results extend directly in the presence of persistent productivity shocks, although they require the additional assumption that the treatment is orthogonal to long-term productivity.

3.2 Aggregating the Treatment

Our goal is to aggregate the firm-level evidence gathered from the experiment. This aggregation exercise consists of measuring the effect of generalizing the policy from the subset of treated firms to *all* firms in the economy. An alternative version of our aggregation exercise consists of extending the policy to just a larger fraction of firms in the economy. We focus on full aggregation here, and explore partial aggregation in Section 5. We summarize the total aggregation exercise in the paragraph below.

Objective 1 (Aggregation of the Treatment)**.** *The aggregation of the treatment consists of computing the log-change in output and TFP when moving from an economy where* no firms *are non-treated ($\Theta = \Theta_0$ for all firms) to an economy where* all firms *are treated ($\Theta = \Theta_1$).*

Note (w_0, Y_0, TFP_0) (resp. (w_1, Y_1, TFP_1)) the equilibrium quantities in the economy where no firms (resp. all firms) are treated. Then:

$$\Delta \log Y = \log(Y_1) - \log(Y_0) = \frac{\alpha(1+\epsilon)}{1-\alpha}\left(-\Delta\mu_\tau + \frac{1}{2}\frac{\theta}{1-\theta}\left(\alpha\Delta\sigma_\tau^2 - 2\Delta\sigma_{\tau z}\right)\right) \tag{7}$$

$$\Delta \log(TFP) = \log(TFP_1) - \log(TFP_0) = -\frac{\alpha}{2}\left(1 + \frac{\alpha\theta}{1-\theta}\right)\Delta\sigma_\tau^2 \tag{8}$$

where:

$$\begin{aligned}
\Delta\mu_\tau &= \mu_\tau(\Theta_1, w_1, Y_1) - \mu_\tau(\Theta_0, w_0, Y_0) \\
\Delta\sigma_\tau^2 &= \sigma_\tau^2(\Theta_1, w_1, Y_1) - \sigma_\tau^2(\Theta_0, w_0, Y_0) \\
\Delta\sigma_{z\tau} &= \sigma_{z\tau}(\Theta_1, w_1, Y_1) - \sigma_{z\tau}(\Theta_0, w_0, Y_0)
\end{aligned}$$

The equations above make clear the challenge faced by the econometrician. The econometrician does not directly observe the distribution of wedges and productivities *conditional* on the new equilibrium (w_1, Y_1), since this new equilibrium is by definition not observed. Therefore, empirically, the econometrician cannot directly estimate $\Delta\mu_\tau$ using the available evidence, but instead can only estimate the following statistic $\widehat{\Delta\mu_\tau}$, which is a priori not equal to $\Delta\mu_\tau$:

$$\begin{aligned}
\widehat{\Delta\mu_\tau} &= \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 1\right) - \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 0\right) \\
&= \mu_\tau(\Theta_1, w_0, Y_0) - \mu_\tau(\Theta_0, w_0, Y_0) \\
&\neq \Delta\mu_\tau = \mu_\tau(\Theta_1, \boldsymbol{w_1}, \boldsymbol{Y_1}) - \mu_\tau(\Theta_0, w_0, Y_0) \quad \text{a priori}
\end{aligned}$$

To perform the aggregation exercise stated in Objective 1, however, the econometrician needs to estimate $\Delta\mu_\tau$ and not $\widehat{\Delta\mu_\tau}$. Of course, this problem is not specific to the mean but is relevant for all moments of the joint distribution of $(z, \log(1+\tau))$. The next Section presents sufficient conditions under which these statistics are in fact similar.

3.3 Scale Invariance of the Wedge Distribution

This section presents one of our main results: we provide sufficient conditions under which the joint distribution of wedges and productivity does not depend on the equilibrium quantities w and Y. As we detail below, the assumptions necessary to obtain this result are satisfied in a large class of models of firm dynamics, commonly used in macro-finance.

Proposition 2 (Distribution of wedges).
Let $S = \frac{Y}{w^{\frac{(1-\alpha)\phi}{(1-\phi)\alpha}}}$ be the steady state "scale" of the economy. Assume that:

1. *(1) the adjustment cost $\Gamma()$, (2) taxes $\mathcal{T}()$, (3) the vector of funding constraint $\boldsymbol{M}()$ and, (4) the equity issuance cost function $\mathcal{C}()$*all *satisfies the following property:*

$$\forall \ (z, \boldsymbol{x}; \boldsymbol{\Theta}, w, Y), \quad Q(z, \boldsymbol{x}; \boldsymbol{\Theta}, w, Y) = S \times Q\left(z, \frac{\boldsymbol{x}}{S}; \boldsymbol{\Theta}, 1, 1\right) \tag{9}$$

2. *The interest rate function $r()$ satisfies the following property:*

$$\forall \ (z, \boldsymbol{x}; \boldsymbol{\Theta}, w, Y), \quad r(z, \boldsymbol{x}; \boldsymbol{\Theta}, w, Y) = r\left(z, \frac{\boldsymbol{x}}{S}; \boldsymbol{\Theta}, 1, 1\right)$$

3. *The survival set $Z()$ does not depend on aggregate conditions:*

$$Z(k', b'; \boldsymbol{\Theta}, w, Y) = Z(k', b'; \boldsymbol{\Theta}, 1, 1)$$

Then, the joint-distribution of z and τ does not depend on (w, Y):

$$F(z, \tau; \boldsymbol{\Theta}, w, Y) \equiv F(z, \tau; \boldsymbol{\Theta})$$

Proof. See Appendix A.3 □

This proposition shows that, given parameters Θ, the ergodic distribution of capital wedges does not depend on the scale of the economy. It is the key result of the paper. It implies that under the assumptions highlighted in Proposition 2, the wedge distribution observed for a subset of firms operating under parameters Θ does not depend on the behavior of other firms, in particular on the parameters $\tilde{\Theta}$ faced by other firms in the economy. This result is necessary to be able to estimate the aggregate counterfactual presented in Objective 1 using firm-level evidence.

As shown in the proof, this result rests on two key assumptions. The first, crucial, assumption is that firm-level production follows a Cobb-Douglas technology, since the multiplicative property of Cobb-Douglas technology ensures that firm-level operating profits scale proportionally to S:

$$\pi\left(z, k; Y, w\right) = \max_{l} Y^{1-\theta} k^{\theta\alpha} l^{\theta(1-\alpha)} - wl = S\pi\left(z, \frac{k}{S}; 1, 1\right)$$

this property is for instance not satisfied by a CES technology as we show in Section 5.5. It arises because of the multiplicative structure of Cobb-Douglas and does not have a deep economic interpretation. The second key set of assumptions is that the frictions or homogeneous of degree 1: If the economy is scaled up by a factor S, frictions have to scale up by the same amount. Taken together, these two assumptions ensure that cash-flows and constraints are scaled up by S. The contraction mapping theorem does the rest: The firm's value function is also scaled up by S which guarantees that all firm policies (debt, capital stock, employment or revenues) are themselves scaled by S. Wedges, which are simple ratios of revenue to capital, are thus invariant with S.

As it turns out, the assumptions under which proposition 2 holds cover a large class of models in the literature on corporate investment. In Section 4, we explain why well-known models satisfy our homogeneity assumptions. We also implement a systematic review of the recent investment literature (which emphasizes various forms of frictions) and find that our assumptions encompass an overwhelming share of existing models in structural corporate finance and macro-investment literatures.

3.4 Taking Stock: Aggregation Formulas

In this Section, we summarize our aggregation methodology. Under the assumptions of Proposition 2, the econometrician can proceed in two steps. First, she uses the firm-level evidence to measure the treatment effect on the three moments introduced in Proposition 1. For instance, focusing on the average log wedge, the econometrician can estimate:

$$\begin{aligned}
\widehat{\Delta\mu_\tau} &= \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=1\right) - \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=0\right) \\
&= \mathbb{E}\left(\log\left(1+\tau\right)|T_i=1\right) - \mathbb{E}\left(\log\left(1+\tau\right)|T_i=0\right) \\
&= \mu_\tau(\Theta_1, w_0, Y_0) - \mu_\tau(\Theta_0, w_0, Y_0) \\
&= \mu_\tau(\Theta_1, w_1, Y_1) - \mu_\tau(\Theta_0, w_0, Y_0) \\
&= \Delta\mu_\tau
\end{aligned}$$

To go from the third to the fourth line in the previous equation is valid under the assumptions of Proposition 2, as the distribution of τ is then independent of w and Y. As a result, the empirical estimate $\widehat{\Delta\mu_\tau}$ now provides us with the relevant statistics to be used in the aggregation exercise. The same reasoning applies to the two other moments of Proposition 1, which can be estimated through:

$$\widehat{\Delta\sigma_\tau^2} = \text{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 1\right) - \text{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 0\right) = \Delta\sigma_\tau^2$$
$$\widehat{\Delta\sigma_{z\tau}} = \text{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i = 1\right) - \text{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i = 0\right) = \Delta\sigma_{z\tau}$$

In a second step, the econometrician then plugs these three statistics into the aggregation formulas (7-8). They require prior knowledge of only three parameters: the capital share α, the extent of competition θ and the labor supply elasticity ϵ. With these 3 parameters only, the econometrician can estimate an aggregate counterfactual which does not require a full structural estimation of the firm-level problem 2 nor a precise mapping from the treatment to the parameters Θ affected by the experiment.

A standard calibration of these parameters in the aggregation formula (7-8) is $\alpha = .33$ (Bartelsman et al. (2013), $\theta = .8$ (Broda and Weinstein (2006)) and $\epsilon = .5$ (Chetty (2012)). With this calibration, the aggregation formula imply that a policy that increases investment by about 2% at the firm-level and leaves the variance of wedges as well as the correlation of wedges with productivity unchanged would increase output by .63%.[5] A policy that would leave the average wedge and the correlation of wedge and TFP unchanged, but would reduce the dispersion of wedges by 1.29 percentage points would similarly lead to an increase in aggregate output of .63%.

4 Relation with Standard Models of Firm Dynamics

Proposition 2 requires that the frictions faced by firms be homogeneous of degree 1. In this section, we first explain how standard models of firm dynamics with frictions often satisfy these assumptions. We discuss in turn real frictions (adjustment cost), financial frictions and taxes. We conclude with a more extensive review of the recent literature on firms dynamics. Among the 44 papers we discuss, our assumptions are satisfied between 80 and 90% of the time.

[5] The revenue to capital ratio is given by: $\Delta \log \frac{py}{k} = -\frac{1-\theta}{1-\theta(1-\alpha)}\Delta \log k$, so that a 2% increase in investment corresponds to a .85% decline in the sales-to-capital ratio ($\Delta\mu_\tau = -.85\%$), which in turn leads to an increase in aggregate output of $\frac{\alpha(1+\epsilon)}{1-\alpha} \times .85\% = .63\%$.

4.1 Adjustment Costs

Consider first the case of adjustment costs. Quadratic adjustment costs to capital, linear adjustment costs, fixed costs that scale either with production, output and capital or discount for capital resale all satisfy the assumptions in Proposition 2. For instance, if $\Gamma()$ is given by:

$$\Gamma(z, \boldsymbol{x}; \Theta, w, Y) = \gamma_1 \frac{(k' - (1-\delta)k)^2}{k} + \gamma_2 k + \mathbb{1}_{\{k'-(1-\delta)k \neq 0\}} (\gamma_3 y + \gamma_4 py + \gamma_5 k) + \gamma_6 k \mathbb{1}_{\{k'-(1-\delta)k<0\}},$$

then, since $y(z, k; \Theta, w, Y) = S \times y(z, \frac{k}{S}; \Theta, 1, 1)$ and $py(z, k; \Theta, w, Y) = S \times py(z, \frac{k}{S}; \Theta, 1, 1)$, it is trivial to show that $\Gamma(z, \boldsymbol{x}; \Theta, w, Y) = S \times \Gamma(z, \frac{\boldsymbol{x}}{S}; \Theta, 1, 1)$

4.2 Financing Frictions

Second, consider the financing side of the model. Our formulation encompass standard models of financing constraints and investment.

Let us start with the interest rate function. For instance, in Michaels et al. (2016) or Gilchrist et al. (2014), debt is risky and in the event that the firm is unable to repay, the lender can seize a fraction $1 - \zeta$ of the firm's fixed assets k. The firm's future market value is not collateralizable, so that a firm's access to credit is mediated by a net worth covenant, which restrains the firm's ability to sell new debt based on its current physical assets and liabilities. Concretely, default is triggered when net worth reaches 0, which defines a threshold value for productivity $\hat{z}$ such that:

$$0 = \kappa_0 S^{1-\phi} e^{\frac{\phi}{\alpha}\hat{z}} k^{\phi} - b + c^k (1-\delta)k, \tag{10}$$

where c_k is the second-hand price of capital, which we treat as a technological parameter. As in Michaels et al. (2016), the right side of the previous equation represents the resources that the firm could raise in order to repay its debt just prior to bankruptcy, which is why its capital is valued at the second-hand price c_k. The wage bill is absent from the previous equation because labor is paid in full, even if the firm subsequently defaults. Finally, the face value of debt discounted at the interest rate $r(z, \boldsymbol{x}; \boldsymbol{\Theta}, w, Y)$ must equal the debt holder's expected payoff discounted at the risk-free rate:

$$\frac{1}{1+r}\left[\int_0^{\hat{z}}\left(\kappa_0 S^{1-\phi}e^{\frac{\phi}{\alpha}z'}k'^{\phi}+(1-\zeta)(1-\delta)k'\right)dH(z'|z)+(1-H(\hat{z}|z))b'\right]=\frac{b'}{1+r(z,\boldsymbol{x};\boldsymbol{\Theta},w,Y)} \tag{11}$$

Equations (10) and (11) provides the joint definition for the interest rate function $r(z,\boldsymbol{x};\Theta,w,Y)$, which satistfies the assumption in Proposition 2. Note first that equation (10) can be rewritten as: $0=\kappa_0 e^{\frac{\phi}{\alpha}\hat{z}}\frac{k}{S}^{\phi}-\frac{b}{S}+c^k(1-\delta)\frac{k}{S}$. As a result, it is clear that $\hat{z}(k,b;\Theta,w,Y)=\hat{z}(\frac{k}{S},\frac{b}{S};\Theta,1,1)$. Also, we can rewrite Equation (11) as:

$$\frac{1}{1+r}\left[\int_0^{z(\frac{k'}{S},\frac{b'}{S};\Theta,1,1)}\left(\kappa_0 e^{\frac{\phi}{\alpha}z'}\frac{k'}{S}^{\phi}+(1-\zeta)(1-\delta)\frac{k'}{S}\right)dH(z'|z)+(1-H(\hat{z}|z))\frac{b'}{S}\right]=\frac{\frac{b'}{S}}{1+r(z,\boldsymbol{x};\boldsymbol{\Theta},w,Y)},$$

so that $r(z,\boldsymbol{x};\boldsymbol{\Theta},w,Y)=r(z,\frac{\boldsymbol{x}}{S};\boldsymbol{\Theta},1,1)$.

Similarly, the specification of debt renegotiation in Hennessy and Whited (2007) would also satisfy these assumptions. More generally, these models make the probability of default independent of the scale of the economy S, and the loss given default proportional to S. These properties ensure our assumption about $r()$ in Proposition 2 is satisfied. Obviously, models of risk-free debt, such as Midrigan and Xu (2014), also satisfy our assumption.

Our assumption on the cost of equity is also verified in Michaels et al. (2016) and Gilchrist et al. (2014), who posit that equity issuances are subject to an underwriting fees such that there is a positive marginal cost to issue equity:

$$\mathcal{C}(z,\boldsymbol{x};\Theta,w,Y)=\lambda|e(z,\boldsymbol{x};\Theta,w,Y)|\mathbb{1}_{\{e(z,\boldsymbol{x};\Theta,w,Y)<0\}}$$

Given that $e(z,\boldsymbol{x};\Theta,w,Y)=Se(z,\frac{\boldsymbol{x}}{S};\Theta,1,1)$, it is obvious that $\mathcal{C}(z,\boldsymbol{x};\Theta,w,Y)=S\times\mathcal{C}(z,\frac{\boldsymbol{x}}{S};\Theta,1,1)$. Thus, the financing frictions specified in Gilchrist et al. (2014) and Michaels et al. (2016) satisfy the assumptions of Proposition 2. Additionally, it is obvious to see that fixed or quadratic issuance costs would satisfy our assumptions as long as they are appropriately scaled with the size of the firm. For instance, $\psi\frac{e_{it}^2}{k_{it}}\mathbb{1}_{e_{it}<0}$, or $\iota k_{it}\mathbb{1}_{e_{it}<0}$ would fall in this category.

Finally, our formulation of financing frictions also encompasses debt constraints as for instance in Midrigan and Xu (2014) or Catherine et al. (2017). In Midrigan and Xu (2014), debt is assumed to be risk-free through full collateralization: $b'\leq\xi k'$ so that $r(z,\boldsymbol{x};\Theta,w,Y)=r_f$ and producers can only issue claims to a fraction χ of their future profits: $e(z, \boldsymbol{x}; \Theta, w, Y) \geq -\chi V(z, \boldsymbol{x}; \Theta, w, Y)$. In this case, the vector $\boldsymbol{M}(z, \boldsymbol{x}; \Theta, w, Y)$ consists of the last two inequalities, and it is direct to see that both $\boldsymbol{M}$ and $r()$ satisfy the assumptions of Proposition 2. Of course, any combination of the constraints in Midrigan and Xu (2014) and Hennessy and Whited (2007) would also satisfy these assumptions. Note also that our model also encompasses debt constraints where debt financing is limited by existing or future cash flows ($b \leq \iota e(z, \boldsymbol{x}; \Theta, w, Y)$).

4.3 Taxes

Standard specifications for the corporate income tax satisfy the assumption of Proposition 2: $\mathcal{T}(z, \boldsymbol{x}; \Theta, w, Y) = \tau \max(0, \pi(z, \boldsymbol{x}; \Theta, w, Y) - \delta k - b)$. However, a progressive tax system would violate our assumptions.

4.4 Recent literature review

In this section, we extend the above analysis from a few examples to 44 papers taken in the recent literature on firm dynamics. There are many papers in this space so we had to restrict ourselves to a subsample. Our sampling method was designed to overweight recent and cited papers in this literature, it does not aim to be exhaustive. The sample was constructed with the following process: Start from all papers citing Hennessy and Whited (2007), Midrigan and Xu (2014) and Moll (2014). Then require that they were published in the top 3 finance outlets, the top 5 economics journals plus AEJs and JME. Finally, further restrict to papers have at least 50 google scholar citations.

We end up with 44 papers. For each of these papers, we check 5 of our assumptions and report the results of this review in Table 1. Modelling choices made in these papers are almost always consistent with our assumptions. Column 1 verifies if the production function is Cobb-Douglas or not. In *all* papers but one, the Cobb-Douglas assumption is valid. In 5 papers, the authors however added to Cobb-Douglas a non-scalable fixed cost to model operating leverage. That particular dimension does not fit our assumptions and would prevent the results from proposition 2 from holding. In column 2, we look at adjustment costs. In almost all papers, physical adjustment costs (real frictions) satisfy our assumptions. This happens because, even when they are fixed (not a function of the amount of investment), they are still typically scaled by total sales, which satisfies our assumptions as decribed in Section 4.1. Columns 3 and 4 focus on financing frictions. The borrowing constraint is almost always (in 88% of the paper that have such a constraint) scale free, which is in line with our assumptions. The equity issuance costs violate them more often: In 9 models (out of 44), there is a fixed equity issuance cost that does not scale with the economy. This is consistent with the standard view in corporate finance, but only one macro model has this feature though. Finally, taxes are also overwhelmingly consistent withour assumption, with corporate tax rates being constant for firms making positive profits and zero for firms making negative accounting profits. Only 2 model deviate from this, essentially because they assume progressive taxe rates on firms. Overall, existing models in the contemporary literature satisfy the assumptions of Proposition 2.

5 Robustness and Extensions

This section proposes extensions to the methodology developed in Section 3 and the model presented in Section 2. The first four subsections investigate the effect on our formulas of increasing heterogeneity in frictions, treatment, production and competition. The last one analyzes the effect of deviatinig from the Cobb-Douglas framework. Additional results are described in Appendix B to conserve space.

5.1 Productivity Heterogeneity

In this section, we explore the effect of unobserved heterogeneity on aggregation. To do so, we augment the baseline model with heterogeneous long-term productivity levels. Indeed, in the baseline model, all firms are identical in the steady state because their productivities are drawn fom the same distribution. Hence, in the baseline model, any group of firms is representative of all the firms, so data from the treatment is by design always informative about the steady state of all firms in the economy. But in reality of course, the validity of the experiment can be an issue if the econometrician scales up the experiment to a group of firms that differs from the group of firms treated in the data. In this Section, we explore what happens if firms only differ through long term (unobservable) productivity levels (as for instance in Midrigan and Xu (2014)). We defer the discussion of technology, competition and treatment heterogeneity to the following sections.

THe baseline model is unchanged except that now log productivity is given by $z_{it} + z_i^*$, where z_{it} is the same Markovian process as in the baseline model, while z_i^* is the fixed component, which differs across firms. In this setting, it is straightforward to show that, even if the treatment is not representative of the entire economy, our baseline formulas continue to apply. This result is summarized in the following proposition:

Proposition 3. *Assume firms have heterogeneous long-term productivities z_i^* in addition to Markovian temporary noises. A researcher observes the effect of treatment on the distribution of empirical wedges for a subset of treated firms for which $T_i = 1$. Then, even if T_i and z_i^* are correlated (i.e. the treatment is not representative of the entire economy), the sufficient statistics and aggregation formulas of the baseline are valid.*

Proof. See Appendix A.4. □

This result relies on the Cobb-Douglas technology. The intuition is that, in our setting, sales, employment and capital stock are all multiplied by long-term productivity $e^{z_i^*}$. Hence, the distribution of capital wedges is independent of the distribution of long-term productivities. Hence, our aggregation formulas apply and even when the treatment does not have the same long-term productivity distribution as the scaling up subset, the sufficient statistics on wedges and temporary productivity shocks estimated in the treatment are valid in the scaling-up exercise. This convenient result happens because, in a sense, the firm fixed effect is "embedded" in the empirical wedges, so that their distribution and hence aggregation are unaffected by them.

5.2 Heterogeneous Treatment Effect & Representativeness

In this section, we allow parameters Θ and treatment effects to differ across firms. This leads to slightly modified formulas. This extension allows us to investigate two important situations. First, in many cases, the treatment is continuous instead of homogeneous (for instance, as in Rajan and Zingales (1998) or more recently in Ponticelli and Alencar (2016)). Second, we can also show how our formula apply when treated firms are not representative of the entire economy but only of a subset of firms .

To focus on the pure effect of heterogeneous treatment, we allow treatment to have varying effects, but keep the same simple industry structure as in the baseline. All firms experience the same AR1 productivity shocks, have the same capital share α, and face the same demand elasticity θ. Parameters governing frictions may however vary across firms. To make exposition simple, we partition the economy S into groups s, within which pre- and post-treatment parameters are identical: $S = \cup_s s$. Think of these groups as industries or geographic areas where initial frictions and treatment effect are homogeneous. We note $\Theta_{s,0}$ and $\Theta_{s,1}$ the value of friction parameters in group s before and after the treatment.

Let us note $S_T \subset S$ the subset of groups for which we observe a treatment effect, i.e. a shift in the distribution of empirical wedges attributed to the treatment. In such a setting, it is impossible to generalize the experiment to groups $s' \notin S_T$, since the treatment effect is unknown for these groups. The only meaningful aggregation exercise therefore consists of computing the effect on output and TFP of treating all of the firms in each subset $s \in S_T$. The following proposition describes formulas to that effect:

Proposition 4. *Let S_T be the set of groups for which treatment is observed on at least part of the firms.*

1. *Assume the data allows the researcher to observe the effect of the treatment on empirical wedges for each $s \in S_T$. We compute, for each one of these industries, the following three statistics:*

$$\widehat{\Delta\mu_\tau}(s) = \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 1, i \in s\right) - \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 0, i \in s\right)$$
$$\widehat{\Delta\sigma^2_\tau}(s) = \mathrm{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 1, i \in s\right) - \mathrm{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 0, i \in s\right)$$
$$\widehat{\Delta\sigma_{z\tau}}(s) = \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i = 1, i \in s\right) - \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i = 0, i \in s\right)$$

2. *Define S_A the "aggregation subset", i.e. the list of groups s' for which the three sufficient statistics can be computed using data from the treatment. By definition, $S_A \subseteq S_T$. But the researcher may decide to make the assumption that the empirical statistics above can be used for groups "similar" to some treated groups. Then, by construction of S_A, the researcher can compute the three statistics above $\forall s \in S_A$.*

3. *Then, the aggregate effect on output of treating all firms $s \in S_A$ is given by:*

$$\Delta \log(Y) = \frac{(1+\epsilon)\alpha}{1-\alpha}\sum_s \gamma_s \left(-\Delta\mu_\tau(s) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(-2\Delta\sigma_{z\tau}(s) + \alpha\Delta\sigma^2_\tau(s)\right)\right)$$
$$+ \frac{1}{2}\frac{(1+\epsilon)\alpha}{1-\alpha}\left(\frac{\theta\alpha}{1-\theta}\right)\boldsymbol{var}^*_{\gamma_s}\left(\Delta\mu_\tau(s)\right)$$

where $\gamma_s = \frac{\int_{i\in s} p_i^0 y_i^0 di}{Y^0}$ is the sales share of group s in the absence of treatment if $s \in S_A$; otherwise, $\gamma_s = 0$.

The aggregate effect on TFP is given by:

$$\Delta \log(TFP) = -\frac{\alpha}{2}\left(1+\frac{\alpha\theta}{1-\theta}\right)\sum_s \kappa_s \Delta\sigma^2_\tau(s)$$
$$+\alpha\left(1+\frac{\alpha\theta}{1-\theta}\right)\sum_s(\gamma_s-\kappa_s)\left[-\Delta\mu_\tau(s)+\frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(\alpha\Delta\sigma^2_\tau(s)-2\Delta\sigma_{z\tau}(s)\right)\right]$$
$$+\frac{\alpha}{2}\left(1+\frac{\theta\alpha}{1-\theta}\right)\left(\frac{\alpha\theta}{1-\theta}\boldsymbol{var}_{\gamma_s}\Delta\mu_\tau(s)-\left(1+\frac{\alpha\theta}{1-\theta}\right)\boldsymbol{var}_{\kappa_s}\Delta\mu_\tau(s)\right)$$

where $\kappa_s = \frac{\int_{i\in s}k_i^0 di}{K^0}$ *is the pre-treatment capital share of group* s *if* $s \in S_A$*. If* $s \notin S_A$*,* $\kappa_s = 0$.

Proof. See Appendix A.5. □

These two formulas are easy to interpret. The output formula contains two terms. The first term is simply the weighted average of within-group production effect. The weights are just the share of each group in total sales prior to treatment. Within-group effects have exactly the same formula as the baseline model, but applied at the group level. The second term accounts for the fact that the treatment may reshuffle production across groups. This will increase the variance of output effects across groups, which through the standard log-normal correction increases output.

The TFP formula contains three terms. The first term is simply the weighted average of the treatment effect on each group. Weights are given by pre-treatment capital shares and within group TFP effects are the same as the baseline formula, but applied at the group level. The second and third terms account for the fact that the treatment reshuffles output across groups. The second term contributes positively when the treatment promotes relatively more groups that have a capital share smaller than their sales share (groups that have too little capital). The third term stands for additional misallocation created by heterogeneous treatment. If the treatment is heterogeneous, it will increase cross-industry distortions, and thus reduce productive efficiency.

Results from Proposition 4 are useful to discuss two important extention of the baseline framework. The first one is the case for heterogeneous treatment, which is pervasive in the literature on finance and growth since Rajan and Zingales (1998). In these papers, $S_T = S_A$ typically corresponds to all industries in the economy. In this literature, one measures the effect of a treatment (e.g. making the financial system more efficient) using the differential exposure of industries (financial dependence). The formulas in this Section help leveraging such difference in difference methodology to compute the aggregate effect of the treatment, while at the same time embracing the fact that industries are not affected equally by the treatment. We provide in Section 6 an example from the recent literature that uses a similar methodology (Ponticelli and Alencar, 2016). But these formula could also be directly applied to evaluate the aggregate impact implied by the evidence from Rajan and Zingales (1998).

The second important use of proposition 4 are situations where the empiricist only observes treatment effects for a subset of the data, that is not necessarily representative of the entire economy. In this case, a full scale-up of the treatment is impossible since the effect of distortions on treated firms a priori differs from treatment effects on other groups of firms. An example would be the evaluation of a policy designed to ease credit constraints for small firms such as in Lelarge et al. (2010). Since large firms are presumably less constrained, any evidence of treatment effect can only be scaled up to "reasonably" small firms. To visualize the effect of such partial aggregation on the formulas, it can be useful to assume homogeneous effects on the groups of S_A.Then, the aggregate effect of scaling up to S_A is given by:

$$\begin{aligned}
\Delta\log(Y) &= \frac{(1+\epsilon)\alpha}{1-\alpha}\gamma\left(-\Delta\mu_\tau + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(-2\Delta\sigma_{z\tau} + \alpha\Delta\sigma_\tau^2\right)\right) \\
&\quad + \frac{1}{2}\frac{(1+\epsilon)\alpha}{1-\alpha}\left(\frac{\theta\alpha}{1-\theta}\right)\gamma(1-\gamma)\left(\Delta\mu_\tau\right)^2 \\
\Delta\log(TFP) &= -\frac{\alpha}{2}\left(1+\frac{\alpha\theta}{1-\theta}\right)\kappa\Delta\sigma_\tau^2 \\
&\quad + \alpha\left(1+\frac{\alpha\theta}{1-\theta}\right)(\gamma-\kappa)\left[-\Delta\mu_\tau + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(\alpha\Delta\sigma_\tau^2 - 2\Delta\sigma_{z\tau}\right)\right] \\
&\quad + \frac{\alpha}{2}\left(1+\frac{\theta\alpha}{1-\theta}\right)\left(\frac{\alpha\theta}{1-\theta}\gamma(1-\gamma) - \left(1+\frac{\alpha\theta}{1-\theta}\right)\kappa(1-\kappa)\right)\left(\Delta\mu_\tau\right)^2
\end{aligned}$$

where κ and γ are the pre-treatment capital and sales shares of the aggregation subset S_A. When S_A is the entire economy, $\gamma = \kappa = 1$, and we are back to the baseline aggregation formulas. If however the aggregation exercise is partial, $\kappa < 1$ and $\gamma < 1$, and the additional terms appear. These additional terms account for the fact that the partial scale-up may reallocate more output towards firms that have to little capital to start with (in which case the corrective term is positive, $\gamma > \kappa$). If firms in the scale-up group start with too much capital ($\gamma < \kappa$), the correction is negative.

5.3 Input-Output Linkages

In this Section, we allow for heterogeneous industries and input-output linkages. Industries are heterogeneous in terms of price elasticity and treatment effects, and firms in each industry consume inputs produces by all other (potentially treated) sectors. We first describe how we modify the baseline framework, and then show the new aggregation formulas, which end to be quite similar, in spirit, to the baseline formulas.

There are S industries indexed by s. The final good is produced by combining intermediate goods:

$$Y = \Pi_{s=1}^{S} Y_s^{\phi_s} \text{ and } \sum_{s=1}^{S} \phi_s = 1$$

where ϕ_s is the share of each intermediate *goods* in production, and Y_s is the quantity used for final good production. We choose a Cobb-Douglas aggegator here to avoid the accumulation of terms and maintain focus on the effect of the Input-output structure on our aggregation formulas. We explore a more general nested CES case (without input-output structure) in our next robustness check.

There is a second layer of intermediate production: Intermediate *good* (s) is produced using intermediate *inputs* (i, s) that are imperfectly substitutable, like in the baseline:

$$Q_s = \left(\int (q_{is})^{\theta_s} di \right)^{1/\theta_s}$$

where Q_s is the quantity of intermediate *good* s produced. Out of this production, Y_s is used for final good production, and $M_s = Q_s - Ys$ is used to produce intermediate inputs for all firms (i', s').

Input-output linkages are modeled the following way. Production of *input* (i, s) uses capital and labor, as well as potentially all intermediate *goods* s. This is where we introduce the input-output matrix $\Gamma = (\gamma_{us})$ into the production function of each input (i, s) (omitting the time subscript):

$$q_{i,s} = e^{z_i} k_i^{\alpha_s} l_i^{\beta_s} \Pi_{u=1}^{S} (m_{i,s,u})^{\gamma_{us}}$$

where we assume for convenience constant returns to scale, i.e. $\alpha_s + \beta_s + \sum_u \gamma_{su} = 1$. Let $M_u = \sum_{s=1}^{S} \int m_{i,s,u} di$ be the total consumption of intermediate good u by all firms in all industries. Then, market clearing condition on good s writes: $Q_s = Y_s + M_s$.

In this model, like in the previous section, we now assume that a treatment allows us to measure the effect of the treatment in each industry s by comparing treated and untreated firms. As before, this effect is plausibly heterogeneous. Then, the following proposition summarizes what would be the effect of extending the experiment to all firms.

Proposition 5 (Input-output Linkages)**.** *Denote ϕ_s^* the linkage-adjusted industry share, the s^{th} element of the vector defined by $(I-\Gamma)^{-1}\phi$, with ϕ being the vector of input shares in final production, and Γ the input-output matrix (so $(I-\Gamma)^{-1}$ is the Leontieff inverse). $\alpha^* = \sum_s \alpha_s \phi_s^*$ is the linkage-adjusted capital share. We define the TFP of the economy as $TFP = \frac{Y}{K^{\alpha^*}L^{1-\alpha^*}}$.*
Then :

1. *Assume the data allows the researcher to observe the effect of the treatment on empirical wedges for each sector s. We first compute, for each one of these industries, the following three statistics:*

$$\widehat{\Delta\mu_\tau}(s) = \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 1, i \in s\right) - \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 0, i \in s\right)$$
$$\widehat{\Delta\sigma_\tau^2}(s) = \text{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 1, i \in s\right) - \text{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i = 0, i \in s\right)$$
$$\widehat{\Delta\sigma_{z\tau}}(s) = \text{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i = 1, i \in s\right) - \text{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i = 0, i \in s\right)$$

2. *Then, the aggregate effect of treating all firms in the economy is given by the following two formulas:*

$$\Delta \ln Y = \frac{\alpha^*(1+\epsilon)}{1-\alpha^*}\sum_s \frac{\alpha_s\phi_s^*}{\alpha^*}\left(-\widehat{\Delta\mu_\tau}(s) + \frac{1}{2}\frac{\theta_s}{1-\theta_s}\left(\alpha_s\widehat{\Delta\sigma_\tau^2}(s) - 2\widehat{\Delta\sigma_{z\tau}}(s)\right)\right)$$
$$\Delta \log TFP = -\frac{\alpha^*}{2}\sum_s \kappa_s\left(1 + \frac{\alpha_s\theta_s}{1-\theta_s}\right)\widehat{\Delta\sigma_\tau^2(s)} - \frac{\alpha^*}{2}\Delta var_{\kappa_s}\left(\widehat{\Delta\mu(s)}\right)$$
$$+\,\alpha^*\sum_s\left(\frac{\alpha_s\phi_s^*}{\alpha^*} - \kappa_s\right)\left(-\widehat{\Delta\mu(s)} + \frac{1}{2}\frac{\theta_s}{1-\theta_s}\left(\alpha_s\widehat{\Delta\sigma_\tau^2(s)} - 2\widehat{\Delta\sigma_{z\tau}(s)}\right)\right)$$

where $\gamma_s = \frac{\int_{i\in s} p_i^0 y_i^0 di}{Y^0}$ is the sales share of sector s without treatment. $\kappa_s = \frac{\int_{i\in s} k_i^0 di}{K^0}$ is the capital share of sector s.

Proof. See appendix A.6. □

The above formulas are easy to interpret. Let us start with output formula, which is very similar to the baseline formula. It is a simple weighted average of industry-level output effects of the treatment. This is because the Cobb-Douglas aggregation framework shuts down spending shares reallocation across industries. Industry-level output effects have an expression that is very similar to the baseline, the only difference being that the pre-multiplication factor use α^* instead of α. Industry weights are given by $\frac{\alpha_s \phi_s^*}{\alpha^*}$ and over-weight effects on large sectors or capital intensitve sectors (for which the distiortionary effect is larger). The TFP formula has three terms. The first term is close to a weighted average of baseline formulas for each industry. Industry weights are given by the capital share of each sector. The second term is the effect of capital reallocation across industries: If the treatment increases cross-industry distortions, this will have a negative impact on aggregate productivities. If the treatment is homogeneous across sectors, this effect is not present. The third term contributes positively if the treatment increases output more in industries that had "too little capital" before the treatment.

IO linkage effects are embedded in the modified capital share $\alpha^* = \sum_s \alpha_s \phi_s^*$. To visualize the quantitative impact of linkages in this formula, the easiest is to focus on a the version of the model that is closest to the baseline, except for the presence of IO linkages. So we set $\alpha_s = \alpha$, $\beta_s = \alpha$ and $\theta_s = \theta$. Further, we assume, as in the baseline, that the treatment is homogeneous. In this case, the two formulas drastically simplify to:

$$\begin{aligned}
\Delta \ln Y &= \frac{\alpha^*(1+\epsilon)}{1-\alpha^*}\left(-\widehat{\Delta\mu_\tau} + \frac{1}{2}\frac{\theta}{1-\theta}\left(\alpha\widehat{\Delta\sigma_\tau^2} - 2\widehat{\Delta\sigma_{z\tau}}\right)\right) \\
\Delta \log TFP &= -\frac{\alpha^*}{2}\left(1+\frac{\alpha\theta}{1-\theta}\right)\widehat{\Delta\sigma_\tau^2}
\end{aligned}$$

which are nearly identical to the baseline formulas (note that the sufficient statistics are not indexed by s as we are assuming homogeneous treatment effects in this discussion). The only difference is coming from the use of $\alpha^* = \alpha \times 1'\left(I-\Gamma\right)^{-1}\phi$, which is the capital share adjusted for input-output effects. α^* typically depends on the shape of the IO matrix, but the general intuition is that it is bigger when large sectors s are the ones that are also used in intermediate production. When this happens, taking the IO structure of the economy tends to make aggregate effects larger.[6].

[6] A simple way to view this is to further assume that $\Gamma = \frac{\gamma}{S}J$. In this case, $\alpha^* = \frac{\alpha}{1-\gamma}$. As the share of intermediate goods increase, the apparent capital share blows up because of network effects. For instance, assuming $\gamma = .5$ and a full IO matrix, the IO-adjusted TFP effect would be multiplied by 2 compared to the baseline.

5.4 Heterogeneous Mark-ups

In this Section, we explore how our aggregation strategy is affected by allowing for heterogeneous mark-ups. To study this, we use a nested CES framework: Starting form the IO linkage model of the previous section, we take out the linkages, but allow final good aggregation to take a more general CES shape. The reason for this is that CES aggregator allow the treatment to shift output shares which gives a more interesting role to heterogeneous mark-ups. Because the algebra is somewhat more cumbersome, we only discuss here the output formula. The TFP formula is available from the authors upon request.

Compared to the model of Section 5, we now assume that the final good aggregator writes as:

$$Y = \left(\sum_{k=1}^{S} \chi_s Y_s^{\psi} \right)^{\frac{1}{\psi}}, \text{ with } \sum_{s=1}^{S} \chi_s = 1$$

where the case $\psi = 0$ corresponds to the results of Section 5. Allowing for $\psi \neq 0$ creates reallocation of factors across industries and thus give rise to costly heterogeneous mark-ups. As in Section 5, the price-elasticity of demand is allowed to vary by industry:

$$Y_s = \left(\int_{i \in s} y_{is}^{\theta_s} di \right)^{\frac{1}{\theta_s}}$$

We describe the output aggregation procedure in the following proposition:

Proposition 6. *In the framework with heterogeneous mark-ups and general CES aggregator, proposition 2 applies, i.e. the joint distribution of z and τ in industry s does not depend on w nor on $(Y_{s'})_{s' \in [1,S]}$.*

So the aggregation of treatment effects proceeds as follows.

1. *Define industry-level treatment effects:*

$$\widehat{\Delta\mu_\tau}(s) = \mathbb{E}\left(\log\left(\frac{p_{it} y_{it}}{k_{it}} \right) | T_i = 1, s_i = s \right) - \mathbb{E}\left(\log\left(\frac{p_{it} y_{it}}{k_{it}} \right) | T_i = 0, s_i = s \right)$$

$$\widehat{\Delta\sigma_\tau^2}(s) = \text{Var}\left(\log\left(\frac{p_{it} y_{it}}{k_{it}} \right) | T_i = 1, s_i = s \right) - \text{Var}\left(\log\left(\frac{p_{it} y_{it}}{k_{it}} \right) | T_i = 0, s_i = s \right)$$

$$\widehat{\Delta\sigma_{z\tau}}(s) = \text{Cov}\left(\log\left(\frac{p_{it} y_{it}}{k_{it}} \right), z_{it} | T_i = 1, s_i = s \right) - \text{Cov}\left(\log\left(\frac{p_{it} y_{it}}{k_{it}} \right), z_{it} | T_i = 0, s_i = s \right)$$

2. *Extending industry-level treatments to all firms in the industry lead to a change in aggregate output:*

$$
\begin{aligned}
\Delta \ln(Y) = & \frac{(1+\epsilon)\alpha}{1-\alpha} \sum_{s=1}^{S} \gamma_s \left[-\Delta\mu_\tau(s) + \frac{1}{2}\left(\frac{\theta_s}{1-\theta_s}\right)\left(-2\Delta\sigma_{z\tau}(s) + \alpha\Delta\sigma_\tau^2(s)\right)\right] \\
& + \frac{1}{2}\frac{(1+\epsilon)\alpha}{1-\alpha}\left(\frac{\psi\alpha}{1-\psi}\right) Var^{\star}_{\gamma_s}\left(\Delta\mu_\tau(s)\right) \\
& + \frac{\alpha\psi}{1-\psi}\sum_{s=1}^{S}(\gamma_s - w_s)\left[-\Delta\mu_\tau(s) + \frac{1}{2}\left(\frac{\theta_s}{1-\theta_s}\right)\left(-2\Delta\sigma_{z\tau}(s) + \alpha\Delta\sigma_\tau^2(s)\right)\right] \\
& + \frac{1}{2}\left(\frac{\psi\alpha}{1-\psi}\right)^2 \left[Var^{\star}_{\gamma_s}\left(\Delta\mu_\tau(s)\right) - Var^{\star}_{w_s}\left(\Delta\mu_\tau(s)\right)\right]
\end{aligned}
$$

where $\gamma_s = \frac{P_s^0 Y_s^0}{P^0 Y^0}$ *is the industry share of revenue.* $w_s = \frac{\theta_s \gamma_s}{\sum_{s'} \theta_{s'} \gamma'_{s'}}$ *is the mark-up weighted industry share.* $\Delta var_{\omega_s}(\mu(s))$ *is the variance of industry treatment effects, weighted by industry weights* $\omega_s \in \{\gamma_s, w_s\}$.

Proof. See Appendix A.7. □

In this framework, the TFP formula is more complex than in Section 5. To build intuition on the new output formula, it is useful to start with the case of homogeneous mark-ups, i.e. $\theta_s = \theta$ which implies $\gamma_s = w_s$. In this case, only the first two terms are present:

$$
\begin{aligned}
\Delta \ln(Y) = & \frac{(1+\epsilon)\alpha}{1-\alpha} \sum_{s=1}^{S} \gamma_s \left[-\Delta\mu_\tau(s) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(-2\Delta\sigma_{z\tau}(s) + \alpha\Delta\sigma_\tau^2(s)\right)\right] \\
& + \frac{1}{2}\frac{\psi}{1-\psi}\frac{(1+\epsilon)\alpha^2}{(1-\alpha)}\left[Var^{\star}_{\gamma_s}\left(\Delta\mu_\tau(s)\right)\right]
\end{aligned}
$$

This formula is very close to the output formula of the baseline model (7). It contains three moments, two of which are identical to the baseline formula:

$$\Delta \ln(Y) = \frac{(1+\epsilon)\alpha}{1-\alpha}\left[-\widetilde{\Delta\mu_\tau} + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)(-2\widetilde{\Delta\sigma_{z\tau}} + \alpha\widetilde{\Delta\sigma^2_\tau})\right]$$

$$\widetilde{\Delta\mu} = \sum_{s=1}^{S} \gamma_s \Delta\mu_\tau(s)$$

$$\widetilde{\Delta\sigma_{z\tau}} = \sum_{s=1}^{S} \gamma_s \Delta\sigma_{z\tau}(s)$$

$$\widetilde{\Delta\sigma^2_\tau} = \sum_{s=1}^{S} \gamma_s \Delta\sigma^2_\tau(s) + \left(\frac{\psi(1-\theta)}{\theta(1-\psi)}\right) Var^{\star}_{\gamma_s}\left(\Delta\mu_\tau(s)\right)$$

where the first two components are the plain mean and covariance effects, which are exactly identical to the baseline formula. The third component differs from the actual variance of wedges unless $\theta = \psi$. This difference arises from the fact that, when $\psi \neq \theta$, reallocation between industries matters. For instance, when $\psi < \theta$ (inputs are less substituable in final goods production than in intermediate inputs), the aggregate impact on output is less than the baseline formula. This happens because increasing the heterogeneity across industries, when they are not very substitutable, hurts output. Also, note that if $\theta = \psi$, we are back to the baseline formulas as the underlying model is now identical to the baseline.

The bottom line of the above discussion is that the first two terms of the formula in proposition 6 correspond to the case with homogeneous mark-ups. The last two terms therefore correspond to the case with heterogeneous mark-ups. The third term speaks to the effect of treatment *within sectors*. It overweights the overall treatment effect of industries where $\theta_s < \sum_{s'} \theta_{s'}\gamma_{s'}$, i.e. industries whose mark-ups are larger than average. Since firms in these industries are "too small" to start with, reducing wedges in these industries has a stronger aggregate impact than average. The fourth term is about cross-industry reallocation. If the treatment effect is heterogeneous across sectors, it tends to create misallocation (the first part $Var^{\star}_{\gamma_s}\left(\Delta\mu_\tau(s)\right)$), but this effect is attenuated if this heterogeneity is concentrated in high mark-up (low θ_s) sectors. This is because these sectors respond less strongly to the treatment.

5.5 CES Production Function

In this Section, we extend our formulas beyond Cobb-Douglas. We keep the same framework as in Section 2, except that we now assume the production technology to be given by:

$$y_{it} = e^{z_{it}} \left(\alpha k_{it}^{\rho} + (1-\alpha) k_{it}^{\rho} \right)^{\frac{1}{\rho}} \tag{12}$$

where α governs the capital share and ρ is the substitution elasticity between labor and capital. When $\rho = 0$, the model boils down to the derivations in Section 2. But this Section seeks to investigate the robustness of our formulas to the situation where capital and labor are more complement that in Cobb-Douglas technology, which is the consensus in the production function literature (see for instance Oberfield and Raval (2014), who find that, on average, an elasticity of about $-.5$ or equivalenty $\rho = -1 < 0$).

In Appendix A.8, we show that the following proposition holds:

Proposition 7 (CES Production Function)**.** *Assume a small scale empirical treatment. To calculate the effect of extending this empirical treatment to all firms, we can follow the following 4 step approach:*

1. *Compute the following three sufficient statistics from the small-scale experiment:*

$$\widehat{\Delta\mu} = E\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 1\right) - E\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 0\right)$$
$$\widehat{\Delta\sigma_\tau^2} = \text{Var}\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 1\right) - E\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 0\right)$$
$$\widehat{\Delta\sigma_{\tau z}} = \text{Cov}\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 1\right) - E\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 0\right)$$

   where s_i^L is the labor share at the firm level and s_i^K is the capital share.

2. *Compute the following two aggregation sufficient statistics β and γ (not necessarily using the experiment) by running the regression, for firm i in economy c:*

$$\log\left(\frac{s_i^K}{\theta - s_i^L}\right) = cst + \beta \log w_c + \gamma (\log w_c)^2 + u_{ic}$$

   running this regression requires firm level data on labor and capital shares in a cross-section of economies (sectors, cities) and an exogenous source of variation for w.

3. *Compute the additional parameter $a = \frac{s^L}{\theta}$ where s^L is the average labor share in the cross-section. and note $b = 1 - a$.*

4. *Then, the aggregate effect of the experiment is given by:*

$$\Delta \log Y = A\widehat{\Delta\mu} + B\widehat{\Delta\sigma_\tau^2} + C\widehat{\Delta\sigma_{\tau z}} + D(\widehat{\Delta\mu})^2$$

where A, B, C and D are known functions of a,b, β, γ, as well as θ, ϵ and ρ. These functions are reported in Appendix A.8

The first order version of the above formula is much simpler and allows to build intuition:

$$\Delta \log Y = -\frac{1 - \frac{s^L}{\theta}}{\frac{s^L}{\theta} + \beta\left(1 - \frac{s^L}{\theta}\right)}\left(\epsilon + \frac{1}{1-\rho}\right)\Delta E \widehat{\log\left(\frac{s_i^K}{\theta - s_i^L}\right)}$$

Going back to Section 2, the equivalent first order formula in the Cobb-Douglas framework is:

$$-\frac{1 - \frac{s^L}{\theta}}{\frac{s^L}{\theta}}(1+\epsilon)\Delta E \log\left(\frac{py_i}{k_i}\right)$$

.

Compared to Cobb-Douglas, the above formula requires three types of adjustment. First, under CES, as opposed to the Cobb-Douglas case, distiortion are affected by the GE of the economy: This is β. β measures the extent to which expected wedges are affected by the equilibrium wage and creates a correction term. We show in the appendix that wedges are not affected by Y, so this does not need a correction. In the Cobb-Douglas case, we have shown in proposition 2 that $\beta = 0$. But it does not have to be in the CES context, which requires the estimation of additional sufficient statistics to capture the effect of macro variables on distortions.

Second, under CES, general equilibrium effects themselves are more dampening than under Cobb-Douglas if $\rho < 0$. This happens because labor cannot be substituted with capital as easily. This effect appears in the above formula through the term $\epsilon + \frac{1}{1-\rho}$ which is smaller than $1 + \epsilon$. Third, wedges cannot be computed using the sales to capital ratio as in the Cobb-Douglas case: This is the last term, which uses input shares to measure distortions more adequately.

5.6 Additional Results

In Appendix B, we explore additional results. The first extension (Appendix B.1) shows formulas for aggregating treatments designed to increase firm productivity. A few papers (such as for instance Bloom et al. (2013a),Bloom et al. (2013b)) study treatments designed to increase firm-level productivity. Aggregating such micro evidence does however pose the challenge that firms face other frictions that will interfere with scaling the treatment. Our framework allows to take these effects into account in a simple way, for a large set of potential frictions. It highlights that statistics on productivity are not the ones that empirical researchers should collect, but also statistics on varying distortions, as well as correlations between productivity and distortions.

A second extension (B.2) simply explores the effect of decreasing returns to scale and allows for potentially perfect competition. The effect on aggregation formulas is marginal and can safely be neglected as long as technological returns to scale are close to 1. A third extension (B.3) proposes formulas that do not rest on the fact that wedges are small. These formulas invoke different sufficient statistics, that are easy to compute, but less robust to out-of-model heteroskedasticity.

The last two extensions (B.4 and B.5) are devoted to the effect of labor distortions on our baseline formulas. In the first extension, we simply assume that labor distortions are represented by exogenous wedges on wage. This would for instance arise because of firm-varying payroll taxes for instance. This extension makes clear that the empirical researcher needs to retrieve more than the usual dispersion of capital wedges: effect of the reform on the covariance between labor and capital wedge is also a potentially important statistic to inform aggregation. The second extension proceeds to microfound labor wedges through binding firm-level minimum wages on unskilled labor. In this case, the generalization of our approaches requires that the empiricist measures how capital wedges interact with firm-level minimum wages.

6 Two Examples of Application

In this Section, we explain how our methodology could be applied in two recent papers that investigate the causal effect of distortions on firm behavior using well-identified set-ups.

6.1 Reforming Debt Enforcement

The first paper is Ponticelli and Alencar (2016), who studies the effect of court enforcement on the availability of credit to firms, investment, employment and sales growth. In a nutshell, the paper runs the following type of regression, where i is an index for firms and j for a municipality:

$$\Delta y_{ij} = \beta T_j + \epsilon_j \tag{13}$$

where $T_j = \log\left(\frac{\text{backlog}}{\text{judge}}\right)$ is a measure city-level bankruptcy court congestion. Δy_{ij} is the change of firm-level activity (log employment for instance) between before and after a bankruptcy reform. The identification strategy rests on the idea that the reform should have a smaller effect on firms located in cities where courts are congested. So for instance, we would expected β to be negative when looking at corporate investment, as firms benefiting from better debt enforcement benefit from easier access to credit. This is indeed what the paper finds in its Table III.

It would be easy to implement our methodology to estimate the aggregate effect of the reform on TFP and output in all of Brazil. Doing this directly using aggregate data would be impossible, as confounding events may have affected macro variables at the same time. The cross-sectional approach in equation (13) helps with this as it rests on the comparison between cities and therefore filters out aggregate shocks that were hitting the Brazilian economy at the same time. Note that, provided the identification strategy is sound, the treatment T_j is *uncorrelated* with city-level exposure to aggregate shocks. Hence, this approach will be robust to situations where cities may have heterogeneous exposures to aggregate shocks.

Given the heterogeneous treatment setting here, it is natural to use the formulas of Section 5.2. In a first step, researchers would use the paper's firm-level data (which is a large and representative survey of privately held Brazilian firms) in order to calculate the change the wedge distribution around the reform. For each firm i at date t, the log capital wedge is computed as the log ratio of sales to physical assets (Property, Plants and Equipment), or $\log(1+\tau_{it}) = \log\frac{\text{sales}_{it}}{\text{PPE}_{it}}$. Log TFP can be computed as $z_{it} = \frac{1}{\theta}\log\text{sales}_{it} - \alpha\log\text{PPE}_{it} - (1-\alpha)\log\text{Emp}_{it}$. Within each municipality-year, we then compute (1) the empirical mean of $\log(1+\tau_{it})$, (2) the empirical variance of $\log(1+\tau_{it})$ and (3) the empirical covariance between $\log(1+\tau_{it})$ and z_{it}. We then compute the average of these three statistics within city across years, separately before and after the treatment, and differentiate, following the exat same procedure as Ponticelli and Alencar (2016) in order to have just one observation per city. We note these three statistics $\Delta\mu_{\tau,j}$, $\Delta\sigma^2_{\tau,j}$ and $\Delta\sigma_{z\tau,j}$.

Note that we cannot directly inject these statistics into the formulas from Proposition 4, because the city-level shifts in the wedge distribution may come from many possible sources, including city-level exposure to aggregate shocks unrelated to the reform. This is where the empirical strategy (13) comes into play. We run the three regressions:

$$
\begin{aligned}
\Delta\mu_{\tau,j} &= \beta_m T_j + \epsilon_{1,j} \\
\Delta\sigma^2_{\tau,j} &= \beta_v T_j + \epsilon_{2,j} \\
\Delta\sigma_{z\tau,j} &= \beta_c T_j + \epsilon_{3,j}
\end{aligned}
$$

which lead us to the three sufficient statistics needed in Proposition 4:

$$
\begin{aligned}
\widehat{\Delta\mu_\tau}(j) &= \widehat{\beta_m} T_j \\
\widehat{\Delta\sigma^2_\tau}(j) &= \widehat{\beta}_v T_j \\
\widehat{\Delta\sigma_{z\tau}}(j) &= \widehat{\beta}_c T_j
\end{aligned}
$$

which we then plug into the two heterogeneous treatment formulas of Proposition 4, and obtain estimates of the effect of the reform on TFP and output. The advantage of this approach is that it does not require to exactly formulate the structural model of how better debt enforcement affects the financing constraint $M()$, the interest rate function $r()$, or the survival set $Z()$, as long as they satistfy the assumptions of Proposition 2. Another practical advantage of this method is that we can easily obtain confidence bounds on the aggregate impact, through bootstrapping on the firm data set.

6.2 Corporate taxes and economic activity

The second paper is Giroud and Rauh (2016). In this paper, the authors estimate the effect of corporate income tax on corporate outcomes using plant-level U.S. census data and plausibly exogenous variations of plant-level state income tax rates. These variations come from variation in the location of headquarters, and apportionment rules that compute an effective tax rate using sales and employment weights across states. This setting allows them to control for a host of fixed effects (firm-level, state-level, etc.).

Their main regression specification is similar to, for plant p, belonging to firms i at date t:

$$y_{ipt} = \beta t_{it} + X_{ipt} + \epsilon_{ipt}$$

where t_{it} is the firm-level corporate income tax rate, X_{ipt} are the controls including fixed effects and ϵ_{ipt} is the residual, which, conditional on controls, is assumed to be uncorrelated with the tax rate (the empirical setting is argued to be valid).

Such an empirical setting, along with our baseline formulas from Proposition 1, can be leveraged in order to compute the aggregate effect on output and TFP of increasing the corporate income tax rate from 0% to, say, 20%. Since this is a uniform treatment, the baseline formulas of proposition 1 can be used, using the following two-step approach. The first step consists of computing the relevant sufficient statistics of proposition 1. Using the Annual Survey of Manufacture (used in the paper), the researcher would first need to compute the plant-level wedge and productivity. For each plant p, in firm i at date t, the log capital wedge is computed as the log ratio of shipments to capital stock, or $\log(1+\tau_{ipt}) = \log\frac{\text{sales}_{ipt}}{\text{Capital}_{ipt}}$. Log TFP can be computed as the Solow residual $z_{ipt} = \log \text{Shipment}_{ipt} - \alpha \log \text{Capital}_{ipt} - (1-\alpha)\log \#\text{ Employees}_{ipt}$. Then, the researcher would need to run three plant-level regressions:

$$\begin{aligned}
\log(1+\tau_{ipt}) &= \beta_\tau t_{it} + X_{ipt} + \epsilon_{1,ipt} \\
\left(\log(1+\tau_{ipt}) - \log\widehat{(1+\tau_{ipt})}\right)^2 &= \beta_{\tau^2} t_{it} + X_{ipt} + \epsilon_{2,ipt} \\
z_{ipt}\left(\log(1+\tau_{ipt}) - \log\widehat{(1+\tau_{ipt})}\right) &= \beta_{z\tau} t_{it} + X_{ipt} + \epsilon_{3,ipt}
\end{aligned}$$

where $\log\widehat{(1+\tau_{ipt})} = \widehat{beta_\tau} t_{it} + X_{ipt}$ is the predicted value from the first regression.

These coefficient lead us to computing the three key statistics needed to implement the baseline formulas:

$$\begin{aligned}
\widehat{\Delta\mu_\tau} &= .2 \times \widehat{\beta_\tau} \\
\widehat{\Delta\sigma^2_\tau} &= .2 \times \widehat{\beta_{\tau^2}} \\
\widehat{\Delta\sigma_{z\tau}} &= .2 \times \widehat{\beta_{z\tau}}
\end{aligned}$$

which measure the effect on the distribution of wedges of raising the tax rate from 0 to 20%. Of course, this implementation assumes the effect on first and second moments are linear in t. If the researcher does not believe in such linearity, she can easily implement an approach that is more non parametric (through running non-parametric regressions instead of the linear ones above).

Once the researcher has computed the three sufficient statistics above, she just needs to plug these statistics into the baseline aggregation formulas of Proposition 1. This leads to an estimation of the aggregate TFP and output effect of increasing income tax rates from 0% to 20%. An advantage of this method is that it accounts for some GE effects, but also that it allows for taxes to interact with other frictions such as adjustment costs or financing costs, in a potentially flexible way, without explicitly modelling and estimating these frictions. For instance, it may be argued that taxes, because their impair the ability of firms to hoard equity, indirectly tighten financing constraints (Davila and Hébert, 2018). This effect is accounted for provided financing constraints satisfy the homogeneity assumptions in Proposition 2. Note also that it is easy to estimate confidence bands for these macroeconomic variables through bootstrapping on the base sample – i.e. before running the three regressions on wedges.

7 Conclusion

This paper develops a simple sufficient statistics framework to aggregate well-identified firm-level evidence of policy experiments aiming to reduce frictions faced by firms. The methodology proceeds in two steps: (1) using firm-level data, the econometrician estimates the treatment effect of the policy on moments of the joint distribution of productivity, and capital wedges (2) these treatment effects are applied to *all* firms in a general equilibrium model of firm dynamics with real frictions, financial frictions and taxes. Our approach yields simple aggregation formula, that can easily be estimated in (quasi-)experimental settings. These formula can easily be extended to more complex economies (e.g, allowing decreasing returns to scale or heterogeneous industries) or partial aggregation exercises where all firms do not receive the treatment.

While variants of this methodology have been used in recent applied work, our paper explicits a set of conditions under which such an approach is valid: (1) intermediate inputs are combined with (nests of) CES aggregators (2) production takes place according to a Cobb-Douglas technology combining labor and capital (3) capital adjustment costs, financing frictions and taxes satisfy a type of homogeneity condition. While these assumptions may appear restrictive, they are satisfied by a large class of models commonly used in the macro-finance literature.

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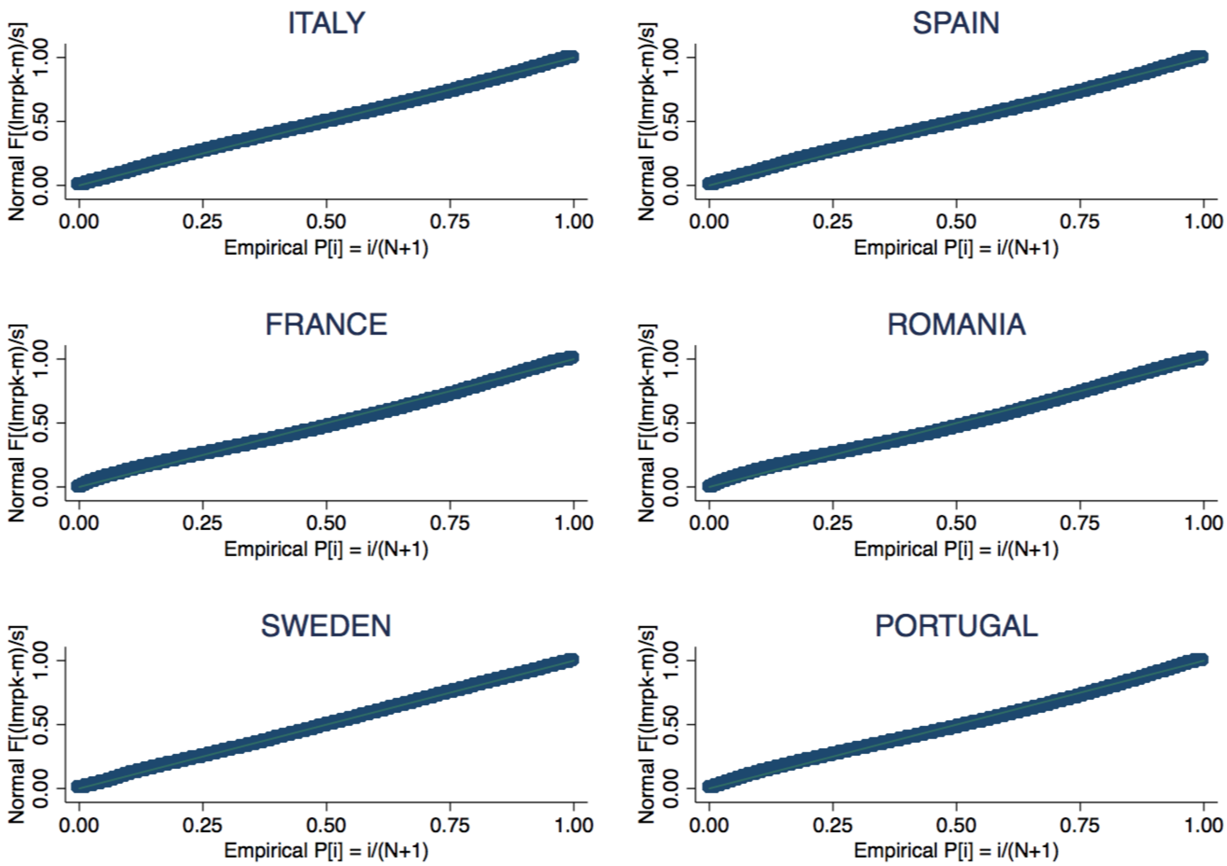
Table 1: Select Literature Review

Authors	Title	Date	(1) Production function function: Cobb-Douglas	(2) Adjustment costs Homogeneous	(3) Borrowing constraint: Homogeneous	(4) Equity issuance: Homogeneous	(5) Taxes schedule: Homogeneous
Panel A: Adjustment Cost papers							
Asker, Collar-Wexler, De Loecker	Dynamic inputs and resource (mis) allocation	2014	Y	Y	-	-	-
Bartelsman, Haltiwanger, Scarpetta	Cross-country differences in productivity: The role of allocation [...]	2013	N	Y	-	-	Y
Basu, Bundick	Uncertainty shocks in a model of effective demand	2017	Y*	Y	-	-	-
Bloom	The impact of uncertainty shocks	2009	Y	Y	-	-	-
Bloom, Bond, Van Reenen	Uncertainty and investment dynamics	2007	Y	Y	-	-	-
Bloom, Floetotto, Jaimovich, Saporta-Eksen, Terry	Really uncertain business cycles	2018	Y	Y	-	-	-
Cooper, Haltiwanger	On the nature of capital adjustment costs	2006	Y	Y	-	-	-
Gourio, Rudanko	Customer capital	2014	Y	Y	-	-	-
Hall	Measuring factor adjustment costs	2004	Y	Y	-	-	-
Peters, Taylor	Intangible capital and the investment-q relation	2017	Y	Y	-	-	-
Panel B: Structural & Dynamic Corporate Finance							
Bolton, Chen, Wang	A Unified Theory of Tobin'sq, Corporate Investment, Financing [...]	2011	Y	Y	-	Y	-
Bolton, Chen, Wang	Market timing, investment, and risk management	2013	Y	Y	-	Y	-
Cummins, Hassett, Oliner	Investment behavior, observable expectations, and internal funds	2006	Y	Y	-	-	-
DeAngelo, DeAngelo, Whited	Capital structure dynamics and transitory debt	2011	Y	Y	N	N	Y
Gamba, Triantis	The value of financial flexibility	2008	Y*	-	Y	Y	N
Gomes, Schmid	Levered returns	2010	Y*	N	Y	N	Y
Hackbarth, Mauer	Optimal priority structure, capital structure, and investment	2011	-	-	Y	N	Y
Hennessy, Whited	Debt dynamics	2005	Y	Y	Y	Y	N
Hennessy, Whited	How costly is external financing? [...]	2007	Y	-	Y	N	Y
Hennessy, Levy, Whited	Testing Q theory with financing frictions	2007	Y	Y	Y	Y	-
Li, Livdan, Zhang	Anomalies	2009	Y*	Y	-	N	-
Liu, Whited, Zhang	Investment-based expected stock returns	2009	Y	Y	-	-	-
Livdan, Sapriza, Zhang	Financially constrained stock returns	2009	Y*	Y	Y	N	-
Miao	Optimal capital structure and industry dynamics	2005	Y	-	Y	-	Y
Michaels, Page, Whited	Labor and capital dynamics under financing frictions	2018	Y	Y	Y	Y	-
Nikolov, Whited	Agency conflicts and cash: Estimates from a dynamic model	2014	Y	Y	Y	Y	Y
Riddick, Whited	The corporate propensity to save	2009	Y	Y	-	N	Y
Whited, Wu	Financial constraints risk	2006	Y	Y	N	N	-
Panel C: Macro-finance with heterogeneous firms							
Arellano, Bai, Zhang	Firm dynamics and financial development	2012	Y	Y	Y	Y	-
Buera, Shin	Financial frictions and the persistence of history [...]	2013	Y	-	Y	-	Y
Buera, Moll	Aggregate implications of a credit crunch [...]	2015	Y	-	Y	-	-
Cooley, Quadrini	Financial markets and firm dynamics	2001	Y	-	Y	Y	-
Fernandez-Villaverde, Guerron-Quintana, Rubio-ramirez, Uribe	Risk matters: The real effects of volatility shocks	2011	Y	Y	-	-	-
Gilchrist, Sim, Zakrajsek	Uncertainty, financial frictions, and investment dynamics	2014	Y	Y	Y	Y	-
Gomes	Financing investment	2001	Y	Y	Y	N	-
Gopinath, Kalemli-Ozcan, Karabarbounis, Villegas-Sanchez	Capital allocation and productivity in South Europe	2017	Y	Y	N	-	-
Gourio	Disaster risk and business cycles	2012	Y	Y	-	-	-
Gourio, Miao	Firm heterogeneity and the long-run effects of dividend tax reform	2010	Y	Y	-	Y	Y
Itskhoki, Moll	Optimal development policies with financial frictions	2018	Y	-	Y	-	Y
Khan, Thomas	Idiosyncratic shocks and the role of nonconvexities in [...] investment	2008	Y	Y	-	-	-
Khan, Thomas	Credit shocks and aggregate fluctuations[...]	2013	Y	Y	Y	Y	-
Liu, Wang, Zha	Landprice dynamics and macroeconomic fluctuations	2013	Y	Y	Y	-	-
Midrigan, Xu	Finance and misallocation: Evidence from plant-level data	2014	Y	Y	Y	Y	-
Moll	Productivity losses from financial frictions[...]	2014	Y	-	Y	-	-
Percent in line with our assumptions in all 44 papers			98%	98%	93%	79%	95%
Percent in line with our assumptions in papers that have the economic force			82%	97%	88%	59%	85%

Table 1: Literature Review (Continued)

Note: This table checks the validity of our assumption in a select review of 44 recent papers from the literature on firm dynamics. We restrict ourselves to all papers citing Hennessy and Whited (2007), Midrigan and Xu (2014) and Moll (2014), published in the top 3 finance outlets, the top 5 economics journals plus AEJs and JME. To further restrict the scope of the review, we asked that the papers have at least 50 google scholar citations. We end up with 44 papers, which we classify in 3 broad strands of literature: "adjustment cost papers", in which adjustment costs are the only friction, papers using dynamic corporate finance models (some of them corresponding to structural estimate, some of them being pure theoretical contributions) and macro-finance paper with financing frictions as well as competitive equilibrium modelling. For each of these papers, we then report if our core assumptions are satisfied: Cobb-Douglas production, and homogeneity of taxes, financing and real frictions. We report the results in columns (1)-(5). Y means that the assumption is satisfied, N that it is not. $-$ means that there is no such force in the model (so that our assumption is by default satisfied). Y^* means that the production function is indeed Cobb-Douglas, but the technology also includes *non-scalable fixed costs* of production. In the bottom two lines of the Table, we report the % of papers for which the assumption is satisfied. In the penultimate line, the % is computed among all papers. In the last line, is is computed only *among papers that have the force being in the model*. Hence, in column 3, 88% of the papers that have borrowing constraints satisfy the assumptions of Proposition 2, but this corresponds to 93% of the papers.

Figure 1: Normal probability plot of log-MRPK for firms in Amadeus



Source: BvD AMADEUS Financials, 2014. Note: This figure shows normal probability plots for 6 OECD countries (France, Spain, Italy, Portugal, Romania and Sweden) for the distribution of log-MRPK. Log-MRPK is computed as the ratio of value added (operating revenue minus materials) and total fixed assets.

ONLINE APPENDIX

A Proofs

A.1 Derivation of output and TFP

Let us begin with the output formula. We start from the three equations:

$$\begin{aligned}
\theta(1-\alpha)\frac{p_{it}y_{it}}{l_{it}} &= w \\
\theta\alpha\frac{p_{it}y_{it}}{l_{it}} &= R(1+\tau_{it}) \\
p_{it}y_{it} &= Y^{1-\theta}e^{\theta z_{it}}k_{it}^{\theta\alpha}l_{it}^{\theta(1-\alpha)}
\end{aligned}$$

Among these equations, the first one is the FOC in labor (assumed frictionless). The second and third ones are *definitions*. The second equation is the definition of the capital wedge τ_{it}. The third equation is the definition of output (Dixit Stiglitz combined with Cobb douglas as explained in the main text). Injecting the top two equations into the last one, we obtain:

$$p_{it}y_{it} = Ye^{\frac{\theta z_{it}}{1-\theta}}\left(\frac{\alpha\theta}{R(1+\tau_{it})}\right)^{\frac{\alpha\theta}{1-\theta}}\left(\frac{(1-\alpha)\theta}{w}\right)^{\frac{(1-\alpha)\theta}{1-\theta}}$$

Exploiting the fact that $Y = \int p_{it}y_{it}di$, this leads to:

$$1 = \left(\frac{(1-\alpha)\theta}{w}\right)^{\frac{(1-\alpha)\theta}{1-\theta}}\int e^{\frac{\theta z_{it}}{1-\theta}}\left(\frac{\alpha\theta}{R(1+\tau_{it})}\right)^{\frac{\alpha\theta}{1-\theta}}di$$

hence the market clearing wage is given by:

$$w \propto \left(\int \frac{e^{\frac{\theta z_{it}}{1-\theta}}}{(1+\tau_{it})^{\frac{\alpha\theta}{1-\theta}}}di\right)^{\frac{1-\theta}{(1-\alpha)\theta}}$$

Besides, labor market equilibrium writes:

$$L = \int l_{it}di \propto \int \frac{p_{it}y_{it}}{w}di \propto \frac{Y}{w}$$

noting that $L \propto w^{\epsilon}$, the labor market clearing condition writes:

$$Y \propto w^{1+\epsilon}$$

which, combined with the expression for the market clearing wage gives the formula for output.

Let us now move to the TFP formula. First, note that:

$$TFP = \left(\frac{Y}{L}\right)^{1-\alpha}\left(\frac{Y}{K}\right)^{\alpha}$$

Given the labor market equilibrium written above, $\frac{Y}{w} \propto w$. So we need to compute $\frac{Y}{K}$.
The aggregate capital stock is given by:

$$K = \int k_{it} di \propto \int \frac{p_{it} y_{it}}{1+\tau_{it}} di \propto Y \left(\frac{1}{w}\right)^{\frac{(1-\alpha)\theta}{1-\theta}} \int \frac{e^{\frac{\theta z_{it}}{1-\theta}}}{(1+\tau_{it})^{1+\frac{\theta\alpha}{1-\theta}}} di$$

We then inject expression for $\frac{Y}{L}$ and $\frac{Y}{K}$ into the TFP formula and obtain:

$$TFP \propto w^{(1-\alpha)\left(1+\frac{\alpha\theta}{1-\theta}\right)} \left(\int \frac{e^{\frac{\theta z_{it}}{1-\theta}}}{(1+\tau_{it})^{1+\frac{\theta\alpha}{1-\theta}}} di \right)^{-\alpha}$$

which after injecting the expression for w, leads to the TFP formula.

A.2 Proof of Proposition 1

First, note $\mu_\tau = \mathbb{E}\tau$, $\sigma_\tau^2 = \mathrm{Var}\tau$ and $\sigma_{z\tau} = \mathrm{Cov}(z,\tau)$. Since the distribution of wedges is a function of Θ and the aggregate equilibrium (w, Y), so are these moments, but we omit the dependence to ease notations.

We start with aggregate production (3):

$$\log Y = (1+\epsilon)\frac{1-\theta}{(1-\alpha)\theta} \log \int_{z,\tau} \left(\frac{e^z}{(1+\tau)^\alpha} \right)^{\frac{\theta}{1-\theta}} dF(z,\tau;\Theta,w,Y)$$

Note $\delta_\tau = \log(1+\tau) - \mu_\tau$, and $u = \frac{\theta}{1-\theta}(z - \alpha\delta_\tau)$. Then:

$$\begin{aligned}
\int_{z,\tau} \left(\frac{e^z}{(1+\tau)^\alpha} \right)^{\frac{\theta}{1-\theta}} dF(z,\tau;\Theta,w,Y) &= \mathbb{E}\left(e^{-\frac{\alpha\theta}{1-\theta}\mu_\tau + u} \right) \\
&= e^{-\frac{\alpha\theta}{1-\theta}\mu_\tau u} \\
&\approx e^{-\frac{\alpha\theta}{1-\theta}\mu_\tau} \mathbb{E}\left(1 + u + \frac{u^2}{2}\right) \\
&\approx e^{-\frac{\alpha\theta}{1-\theta}\mu_\tau} \left(1 + \frac{\mathrm{Var}u}{2}\right) \\
&\approx e^{-\frac{\alpha\theta}{1-\theta}\mu_\tau} \left(1 + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)^2 (\sigma_z^2 - 2\alpha\sigma_{z\tau} + \alpha^2\sigma_\tau^2)\right)
\end{aligned}$$

so that:

$$\begin{aligned}
\log \int_{z,\tau} \left(\frac{e^z}{(1+\tau)^\alpha} \right)^{\frac{\theta}{1-\theta}} dF(z,\tau;\Theta,w,Y) &\approx -\frac{\alpha\theta}{1-\theta}\mu_\tau + \log\left(1 + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)^2 (\sigma_z^2 - 2\alpha\sigma_{z\tau} + \alpha^2\sigma_\tau^2)\right) \\
&\approx -\frac{\alpha\theta}{1-\theta}\mu_\tau + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)^2 (\sigma_z^2 - 2\alpha\sigma_{z\tau} + \alpha^2\sigma_\tau^2)
\end{aligned}$$

which leads to the result. Computation of the TFP formula follows the same logic.

A.3 Proof of Proposition 2

Remember that equity issuance / distributions are given by:

$$\begin{aligned} e_{it} &= \frac{\alpha}{\alpha+(1-\alpha)\phi}\left(\frac{(1-\alpha)\phi}{\alpha+(1-\alpha)\phi}\right)^{\frac{1-\alpha}{\alpha}\phi} S_t^{1-\phi} e^{\frac{\phi}{\alpha}z_{it}} k_{it}^{\phi} - (k_{it+1}-(1-\delta)k_{it}) - \Gamma\left(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w_t, Y_t\right) \\ &\quad + \left(\frac{b_{it+1}}{1+r(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w_t, Y_t)} - b_{it}\right) - \mathcal{T}(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w_t, Y_t), \end{aligned}$$

where $S_t = \frac{Y_t}{w_t^{\frac{(1-\alpha)\phi}{\alpha(1-\phi)}}}$. By combining the different assumptions in Proposition 2, we get that:

$$\begin{aligned} e_{it} &= S_t\left(\frac{\alpha}{\alpha+(1-\alpha)\phi}\left(\frac{(1-\alpha)\phi}{\alpha+(1-\alpha)\phi}\right)^{\frac{1-\alpha}{\alpha}\phi} e^{\frac{\phi}{\alpha}z_{it}}\left(\frac{k_{it}}{S_t}\right)^{\phi} - (\frac{k_{it+1}}{S_t} - (1-\delta)\frac{k_{it}}{S_t}) \right. \\ &\quad \left. - \Gamma\left(z_{it}, \frac{\boldsymbol{x}_{it}}{S_t}; \boldsymbol{\Theta}, 1, 1\right) + \left(\frac{\frac{b_{it+1}}{S_t}}{1+r(z_{it}, \frac{\boldsymbol{x}_{it}}{S_t}; \Theta, 1, 1)} - \frac{b_{it}}{S_t}\right) - \mathcal{T}(z_{it}, \frac{\boldsymbol{x}_{it}}{S_t}; \boldsymbol{\Theta}, 1, 1)\right) \end{aligned}$$

Therefore, $e(z_{it}, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w_t, Y_t) = S_t e(z_{it}, \frac{\boldsymbol{x}_{it}}{S_t}; \boldsymbol{\Theta}, 1, 1)$. Since the equity issuance cost $\mathcal{C}()$ also satisfies property 9, the flow variable in the Bellman equation 2 can be rewritten as:

$$e(z, \boldsymbol{x}; \boldsymbol{\Theta}, w, Y) - \mathcal{C}(z, \boldsymbol{x}; \Theta, w, Y) = S \times \left(e(z_{it}, \frac{\boldsymbol{x}_{it}}{S_t}; \boldsymbol{\Theta}, 1, 1) - \mathcal{C}(z_{it}, \frac{\boldsymbol{x}_{it}}{S_t}; \boldsymbol{\Theta}, 1, 1)\right)$$

We now consider the steady-state of this economy: $w_t = w_{t+1} = w$ and $Y_t = Y_{t+1} = Y$. The Bellman equation 2 becomes:

$$V(z, k, b; \boldsymbol{\Theta}, w, Y) = \left| \begin{array}{l} \max\limits_{k', b'} e(z, \boldsymbol{x}; \boldsymbol{\Theta}, w, Y) - \mathcal{C}(z, \boldsymbol{x}; \boldsymbol{\Theta}, w, Y) + \dfrac{\mathbb{E}_{z'}[V(z', k', b'; \boldsymbol{\Theta}, w, Y)|z]}{1+r_f} \\ \boldsymbol{M}(z, \boldsymbol{x}; \boldsymbol{\Theta}, w, Y) \leq 0 \end{array} \right.$$

LEt T be the Bellman operator defined in the paper in equatio (2). Consider the set of functions $\mathcal{F}$ such that for all $(z,k,b;\boldsymbol{\Theta},w,Y)$, $f(z,k,b;\boldsymbol{\Theta},w,Y) = S \times f(z,\frac{k}{S},\frac{b}{S};\boldsymbol{\Theta},1,1)$. If $f \in \mathcal{F}$, then $\mathcal{B}f \in \mathcal{F}$:

$$
\begin{aligned}
Tf(z,k,b;\boldsymbol{\Theta},w,Y) \quad &= \quad \left| \begin{array}{l} \max\limits_{k',b'} S \times \left(e(z,\dfrac{\boldsymbol{x}}{S};\boldsymbol{\Theta},1,1) - \mathcal{C}(z,\dfrac{\boldsymbol{x}}{S};\boldsymbol{\Theta},1,1) + \dfrac{\mathbb{E}_{z'}[f(z',\frac{k'}{S},\frac{b'}{S};\boldsymbol{\Theta},1,1)|z]}{1+r_f} \right) \\ \boldsymbol{M}(z,\dfrac{\boldsymbol{x}}{S};\boldsymbol{\Theta},1,1) \leq 0 \end{array} \right. \\
&= S\times \left| \begin{array}{l} \max\limits_{k',b'} e(z,\dfrac{k}{S},\dfrac{k'}{S},\dfrac{b}{S},\dfrac{b'}{S};\boldsymbol{\Theta},1,1) - \mathcal{C}(z,\dfrac{k}{S},\dfrac{k'}{S},\dfrac{b}{S},\dfrac{b'}{S};\boldsymbol{\Theta},1,1) + \dfrac{\mathbb{E}_{z'}[f(z',\frac{k'}{S},\frac{b'}{S};\boldsymbol{\Theta},1,1)|z]}{1+r_f} \\ \boldsymbol{M}(z,\dfrac{k}{S},\dfrac{k'}{S},\dfrac{b}{S},\dfrac{b'}{S};\boldsymbol{\Theta},1,1) \leq 0 \end{array} \right. \\
&= S\times \left| \begin{array}{l} \max\limits_{k',b'} e(z,\dfrac{k}{S},k',\dfrac{b}{S},b';\boldsymbol{\Theta},1,1) - \mathcal{C}(z,\dfrac{k}{S},k',\dfrac{b}{S},b';\boldsymbol{\Theta},1,1) + \dfrac{\mathbb{E}_{z'}[f(z',k',b';\boldsymbol{\Theta},1,1)|z]}{1+r_f} \\ \boldsymbol{M}(z,\dfrac{k}{S},k',\dfrac{b}{S},b';\boldsymbol{\Theta},1,1) \leq 0 \end{array} \right. \\
&= \quad S \times Tf(z,\frac{k}{S},\frac{b}{S};\boldsymbol{\Theta},1,1)
\end{aligned}
$$

Since the contraction mapping theorem applies and $\mathcal{F}$ is a compact space, this implies that the value function V also belongs to $\mathcal{F}$:

$$V(z,k,b;\boldsymbol{\Theta},w,Y) = S \times V(z,\frac{k}{S},\frac{b}{S};\boldsymbol{\Theta},1,1).$$

The previous equations also show that, in an economy with scale (w,Y), if (k',b') are the optimal policies for a firm with state variable (z,k,b), then $(\frac{k'}{S},\frac{b'}{S})$ are the optimal policies for a firm with state variables $(z,\frac{k}{S},\frac{b}{S})$ and in the economy with scale ($w=1$, $Y=1$). As a result, the ergodic distribution of $\frac{k}{S}$ in the economy (w,Y) is equal to the ergodic distribution of k in the economy $(1,1)$.

Remember that, by definition in the steady-state, capital wedges are equal to:

$$(1+\tau_{it}) = \frac{\alpha\theta}{r_f+\delta}\frac{p_{it}y_{it}}{k_{it}} = \frac{\alpha\phi}{(\alpha+(1-\alpha)\phi)(r_f+\delta)} e^{\frac{\phi}{\alpha}z_{it}} \left(\frac{k_{it}}{S}\right)^{\phi}$$

Since the ergodic distribution of $(\frac{k}{S})$ in the economy (w,Y) is the same as the ergodic distribtion of k in the economy $(1,1)$ and since the distribution of z is independent of (w,Y), this implies that, in the steady state, the distribution of wedges τ_{it} does not depend on (w,Y) and can be written $G(\tau_i;\Theta)$.

A.4 Proof of Proposition 3

Let the productivity of each be the sum of two terms:

$$z_{it} + z_i^*$$

where z_{it} is the same Markovian process as in the baseline model, while z_i^* is the fixed component, which differs across firms.

In this case, it can be shown that the cash-flow function $E()$ is given by:

$$
\begin{aligned}
e_{it} &= E\left(z_{it} + z_i^*, \boldsymbol{x}_{it}; \boldsymbol{\Theta}, w, Y\right) \\
&= e^{\frac{\theta}{1-\theta} z_i^*} E\left(z_{it}, \tilde{\boldsymbol{x}}_{\boldsymbol{it}}; \boldsymbol{\Theta}, w, Y\right)
\end{aligned}
$$

where $\tilde{\boldsymbol{x}}_{\boldsymbol{it}}$ correponds to present and future capital and debt divided by $e^{\frac{\theta}{1-\theta} z_i^*}$. This property rests on the homogeneity assumptions 9, as well as the multiplicativity of the Cobb-Douglas production function.

A consequence of this property is that the entire dynamic firm problem can be scaled by $e^{\frac{\theta}{1-\theta} z_i^*}$, as a result, employment, capital stock and firm sales can be written as, for each firm of long-term productivity z_i^*. This is summarized in the following Lemma:

Lemma A.1. *The following equalities hold:*

$$
\begin{aligned}
k_{it} &= e^{\frac{\theta}{1-\theta} z_i^*} \tilde{k}_{it} \\
l_{it} &= e^{\frac{\theta}{1-\theta} z_i^*} \tilde{l}_{it} \\
py_{it} &= e^{\frac{\theta}{1-\theta} z_i^*} \tilde{py}_{it}
\end{aligned}
$$

where the terms with a tilde correpond to the firm policies when $z_i^* = 0$ (the case investigated in the baseline model).

Proof. The proof works exactly like for proposition 2 (whose proof is reported in Appendix A.3). We first show that the scaling property of the cash-flow function $E()$ translates into the value function, so that for any state variables $(z^* + z, k, b)$:

$$
V(z^* + z, k, b; \boldsymbol{\Theta}, w, Y) = e^{\frac{\theta}{1-\theta} z^*} V(z, \tilde{k}, \tilde{b}; \boldsymbol{\Theta}, w, Y)
$$

where terms with tildes solve the dynamic program of the baseline model (without long-term heterogeneity). This shows the result of the lemma. □

From lemma A.1, it appears very clearly that the wedges ($\frac{py_{it}}{k_{it}}$) are independent of productivity level z_i^*. Hence, results of proposition 2 hold with firm heterogeneity too since wedges do not depend on it. Also, sufficient statistics on the moments of the distribution of empirical wedges are also independent of the distribution of z_i^*'s.

Aggregation formulas and sufficient statistics are unchanged. We only do the proof for output here. Proof for TFP follows the same logic. Start from firm-level sales (omiting time subscript):

$$
p_i y_i \propto \frac{Y}{w^{\frac{\theta(1-\alpha)}{1-\theta}}} e^{\underbrace{\frac{\theta(z_i^* + z_i)}{1-\theta} - \frac{\theta\alpha}{1-\theta}\log(1+\tau_i)}_{= v_i^* + v_i}}
$$

where $v_i^* = \frac{\theta z_i^*}{1-\theta}$ and v_i corresponds to the rest of the expression.

Aggregating the above expression, after injection of the labor market clearing condition $Y \propto w^{1+\epsilon}$, we obtain:

$$\log Y = \frac{(1+\epsilon)(1-\theta)}{\theta(1-\alpha)} \log \int e^{v_i^* + v_i} di + \text{cst}$$

Since v_i^* and v_i are independent, this is equivalent to:

$$\log Y = \frac{(1+\epsilon)(1-\theta)}{\theta(1-\alpha)} \log \int e^{v_i^*} di + \underbrace{\frac{(1+\epsilon)(1-\theta)}{\theta(1-\alpha)} \log \int e^{v_i} di + \text{cst}}_{\text{log output without heterogeneity}}$$

, hence, we can differentiate between before and after scaling up the treatment:

$$\Delta \log Y = \Delta \frac{(1+\epsilon)(1-\theta)}{\theta(1-\alpha)} \log \int e^{v_i} di$$

which exploits the fact that the treatment does not affect the distribution of fixed effects v_i^*. Since the distribution of v_i only depends on the distribution of wedges, the result of the proposition for output is shown.

A.5 Proof of Proposition 4

Output Formula Sales are given by:

$$p_i y_i \propto \frac{Y}{w^{\frac{\theta(1-\alpha)}{1-\theta}}} e^{\overbrace{\frac{\theta z_i}{1-\theta} - \frac{\theta\alpha}{1-\theta}\log(1+\tau_i)}^{=v_i}}$$

and group-level sales:

$$Y_s = \int_{i \in s} p_i y_i di \propto \frac{Y}{w^{\frac{\theta(1-\alpha)}{1-\theta}}} \underbrace{\int_{i \in s} e^{v_i} di}_{=e^{m(s)}}$$

Hence,

$$\log Y = \frac{(1+\epsilon)(1-\theta)}{\theta(1-\alpha)} \log \left(\sum_s e^{m(s)} \right) + \text{cst}$$

Define the Domar weight γ_s before treatment as:

$$\gamma_s = \frac{Y_s}{Y} = \frac{e^{m_0(s)}}{\sum_{s'} e^{m_0(s')}}$$

then, the difference in log production between a situation where all firms are treated and all firms untreated is given by:

$$\Delta \log Y = \frac{(1+\epsilon)(1-\theta)}{\theta(1-\alpha)} \left[\log \left(\sum_s e^{m_0(s) + \Delta m(s)} \right) - \log \left(\sum_s e^{m_0(s)} \right) \right]$$

We now assume that $\Delta m(s) \ll 1$ so that:

$$\begin{aligned}\Delta \log Y &\approx \frac{(1+\epsilon)(1-\theta)}{\theta(1-\alpha)} \log\left(1 + \sum_s \gamma_s \Delta m(s) - \frac{1}{2}\sum_s \gamma_s (\Delta m(s))^2\right) \\ &\approx \frac{(1+\epsilon)(1-\theta)}{\theta(1-\alpha)} \left(\sum_s \gamma_s \Delta m(s) - \frac{1}{2}\textbf{var}_{\gamma_s}(\Delta m(s))\right)\end{aligned}$$

We now need to compute $\Delta m(s)$. We assume small deviation of $v_i = \frac{\theta z_i}{1-\theta} - \frac{\theta(1-\alpha)}{1-\theta}\log(1+\tau_i)$ around its mean both before and after treatment. Assuming the treatment does not affect the distribution of log productivities, we obtain:

$$\begin{aligned}\Delta m(s) &= \Delta \log \int_{i\in s} e^{v_i} di \\ &\approx \Delta \mathbb{E} v(s) + \frac{1}{2}\Delta \textbf{var}\,(v_i) \\ &= \frac{\theta\alpha}{(1-\theta)}\left(-\Delta\mu_\tau(s) + \frac{1}{2}\frac{\theta}{1-\theta}\left(\alpha\Delta\sigma_\tau^2(s) - 2\Delta\sigma_{z\tau}^2(s)\right)\right)\end{aligned}$$

which we plug back into the output formula and obtain:

$$\begin{aligned}\Delta \log(Y) = &\frac{(1+\epsilon)\alpha}{1-\alpha}\sum_s \gamma_s \left(-\Delta\mu_\tau(s) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(-2\Delta\sigma_{z\tau}(s) + \alpha\Delta\sigma_\tau^2(s)\right)\right) \\ &+ \frac{1}{2}\frac{(1+\epsilon)\alpha}{1-\alpha}\left(\frac{\theta\alpha}{1-\theta}\right)\textbf{var}^*_{\gamma_s}\left(\Delta\mu_\tau(s)\right)\end{aligned}$$

We then compute γ_s. We start from the fact that v_i has small deviations around its mean *before* the treatment, so that:

$$m_0(s) = \log \int_{i\in s} e^{v_i} di = n_s\left(\mu_v(s) + \frac{\sigma_v^2(s)}{2}\right)$$

where $\mu_v(s)$ is the expectation of v_i in group s, $\sigma_v^2(s)$ is the variance of v_i in group s and n_s is the measure of group s.

TFP Formula

As usual, start from the fact that:

$$\begin{aligned}\log TFP &= -\alpha \log\frac{K}{Y} + (1-\alpha)\log\frac{Y}{L} \\ &= -\alpha\log\frac{K}{Y} + (1-\alpha)\log w + \textbf{cst}\end{aligned}$$

The second part will come straight from the outputequation. To get the first part, we start from capital demand:

$$k_i = \alpha\theta \frac{p_i y_i}{1+\tau_i} \propto \frac{Y}{w^{\frac{\theta(1-\alpha)}{1-\theta}}} e^{\underbrace{\frac{\theta z_i}{1-\theta} - \left(1+\frac{\theta(1-\alpha)}{1-\theta}\right)\log(1+\tau_i)}_{=h_i}}$$

Aggregating over the entire economy, we obtain:

$$\frac{K}{Y} \propto \frac{1}{w^{\frac{\theta(1-\alpha)}{1-\theta}}} \sum_s e^{n(s)}$$

where $n(s) = \log \int_{i \in s} e^{h_i} di$.

So TFP writes:

$$\begin{aligned}
\log TFP &= -\alpha \log \sum_s e^{n(s)} + \left(\frac{\theta\alpha^2}{1-\theta} + 1 - \alpha\right) \log w \\
&= -\alpha \log \sum_s e^{n(s)} + (1-\alpha)\left(\frac{\theta\alpha}{1-\theta} + 1\right) \frac{1}{1+\epsilon} \log Y
\end{aligned}$$

We exploit the facts that:

$$\Delta \log \sum_s e^{n(s)} \approx \sum_s \kappa_s \Delta n(s) + \frac{1}{2} \textbf{var}_{\kappa_s} \Delta n(s)$$

and:

$$\Delta n(s) \approx \left(1+\frac{\theta\alpha}{1-\theta}\right)\left(-\Delta\mu_\tau(s) + \frac{1}{2}\left(1+\frac{\theta\alpha}{1-\theta}\right)\Delta\sigma_\tau^2(s) - \frac{\theta}{1-\theta}\Delta\sigma_{z\tau}(s)\right)$$

which leads to the formula for TFP:

$$\begin{aligned}
\Delta \log(TFP) = &-\frac{\alpha}{2}\left(1+\frac{\alpha\theta}{1-\theta}\right)\sum_s \kappa_s \Delta\sigma_\tau^2(s) \\
&+ \alpha\left(1+\frac{\alpha\theta}{1-\theta}\right)\sum_s (\gamma_s - \kappa_s)\left[-\Delta\mu_\tau(s) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(\alpha\Delta\sigma_\tau^2(s) - 2\Delta\sigma_{z\tau}(s)\right)\right] \\
&+ \frac{\alpha}{2}\left(1+\frac{\theta\alpha}{1-\theta}\right)\left(\frac{\alpha\theta}{1-\theta}\textbf{var}_{\gamma_s}\Delta\mu_\tau(s) - \left(1+\frac{\alpha\theta}{1-\theta}\right)\textbf{var}_{\kappa_s}\Delta\mu_\tau(s)\right)
\end{aligned}$$

A.6 Proof of Proposition 5

Formula for Output

Because there is perfect competition in the final good market, the demand for industry s bundle coming from the final good market is given by:

$$\phi_s PY = P_s Y_s \Rightarrow Y_s \propto \frac{Y}{P_s},$$

where we have normalized the price of the final good market to 1 ($P = 1$).

Perfect competition in the production of industry bundles leads to the following demand curve for product i in industry s:

$$P_s \left(\frac{q_{is}}{Q_s}\right)^{\theta_s - 1} = p_{is}$$

The first-order condition w.r.t. bundles from industry $j \in [1, S]$ implies that:

$$P_s Q_s^{1-\theta_s} \theta_s \gamma_{js} \left(y_{is}\right)^{\theta_s} = P_j m_{isj}$$

As a result, the total demand for bundle j from firms in industry s simply comes from aggregating the previous equation across all firms i in industry s:

$$\theta_s \gamma_{js} P_s Q_s = P_j \underbrace{\int_i m_{isj} di}_{=M_{sj}},$$

where M_{sj} corresponds to the demand for industry j's bundles coming from industry s.

As a result, the total demand for industry j bundles coming from intermediary inputs, $M_j = \sum_{s=1}^{S} M_{sj}$ is simply:

$$P_j M_j = \sum_{s=1}^{S} \theta_s \gamma_{js} P_s Q_s$$

Remember that the demand for industry j bundles coming from the final good market is Y_j which satisfies $\phi_j Y = P_j Y_j$.

As a result, the total demand for industry j bundle is simply given by:

$$Q_j = M_j + Y_j = \frac{\sum_{s=1}^{S} \theta_s \gamma_{js} P_s Q_s + \phi_j Y}{P_j} \Rightarrow P_j Q_j = \sum_{s=1}^{S} \theta_s \gamma_{js} P_s Q_s + \phi_j Y$$

From the above equation, it appears clearly that aggregate sales $P_s Q_s$ in each industry s are proportional to Y – each with a different proportionality coefficient: $P_s Q_s \propto Y$.

Turning back to the optimisation problem, the labor first order condition for each firm leads to:

$$P_s Q_s^{1-\theta_s} \theta_s \beta_s \left(y_{is}\right)^{\theta_s} = w l_{is}$$

Aggregating across firm i in industry s, then across industries leads to:

$$wL = \sum_{s=1}^{S} \theta_s \beta_s P_s Q_s \propto Y$$

where we use the fact that each PsQ_s is proportional to Y. Given that labor supply is given by $L^s = \bar{L} \left(\frac{w}{\bar{w}}\right)^{\epsilon}$, we obtain that:

$$\boxed{Y \propto w^{1+\epsilon}}$$

which is the first part of our equilibrium solution.

We now need to compute the equilibrium wage. First order conditions in labor, capital, and inputs are given by:

$$\begin{cases} k_{is} = \alpha_s \theta_s \dfrac{p_{is} y_{is}}{R(1+\tau_i)} \\ l_{is} = \beta_s \theta_s \dfrac{p_{is} y_{is}}{w} \\ m_{isj} = \gamma_{js} \theta_s \dfrac{p_{is} y_{is}}{P_j} \end{cases}$$

We can use the last three equations to compute firm i's output:

$$p_{is} y_{is} = e^{\theta_s z_i} P_s Q_s^{1-\theta_s} (p_{is} y_{is})^{\theta_s} \left(\frac{\alpha_s \theta_s}{R(1+\tau_i)} \right)^{\alpha_s \theta_s} \left(\frac{\beta_s \theta_s}{w} \right)^{\beta_s \theta_s} \left[\prod_{j=1}^{S} \left(\frac{\gamma_{js} \theta_s}{P_j} \right)^{\gamma_{js} \theta_s} \right]$$

As a result:

$$p_{is} y_{is} = e^{\frac{\theta_s}{1-\theta_s} z_i} P_s^{\frac{1}{1-\theta_s}} Q_s \left(\frac{\alpha_s \theta_s}{R(1+\tau_i)} \right)^{\alpha_s \frac{\theta_s}{1-\theta_s}} \left(\frac{\beta_s \theta_s}{w} \right)^{\beta_s \frac{\theta_s}{1-\theta_s}} \left[\prod_{j=1}^{S} \left(\frac{\gamma_{js} \theta_s}{P_j} \right)^{\gamma_{js} \frac{\theta_s}{1-\theta_s}} \right]$$

We can aggregate the previous equation across all firms i industry s:

$$P_s Q_s = P_s^{\frac{1}{1-\theta_s}} Q_s \left(\frac{\alpha_s \theta_s}{R} \right)^{\alpha_s \frac{\theta_s}{1-\theta_s}} \left(\frac{\beta_s \theta_s}{w} \right)^{\beta_s \frac{\theta_s}{1-\theta_s}} \left[\prod_{j=1}^{S} \left(\frac{\psi_{js}}{P_j} \right)^{\frac{\gamma_{js} \theta_s}{1-\theta_s}} \right] \underbrace{\int \frac{e^{\frac{\theta_s}{1-\theta_s} z_i}}{(1+\tau_i)^{\alpha_s \frac{\theta_s}{1-\theta_s}}} di}_{=J_s}$$

The previous equation implies that the price of industry s bundles is simply:

$$P_s \propto w^{\beta_s} \left[\prod_{j=1}^{S} (P_j)^{\gamma_{js}} \right] J_s^{-\frac{1-\theta_s}{\theta_s}}$$

Taking the logarithm of the previous equation, we get that:

$$\ln(P_s) = \beta_s \ln(w) - \frac{1-\theta_s}{\theta_s} \ln(J_s) + \sum_{j=1}^{S} \gamma_{js} \ln(P_j) + cst$$

, which we can write, in vector terms (bold letters are the vectors corresponding to the same scalars):

$$\ln(\mathbf{P}) = (I - \Gamma')^{-1} \left(\boldsymbol{\beta} \ln(w) - \frac{\mathbf{1} - \boldsymbol{\theta}}{\boldsymbol{\theta}} \ln(\mathbf{J}) \right) + cst$$

Now, remember that that $\Pi_{s=1}^{S} P_s^{\phi_s} \propto \Pi_{s=1}^{S} \left(\frac{Y_s}{Y}\right)^{\phi_s} \propto$ cst. Hence, in log vector terms, we have that: $\phi' \ln P =$

cst, hence, combining this with the above equation:

$$\boldsymbol{\phi'} \left(I - \Gamma'\right)^{-1} \left(\boldsymbol{\beta} \ln(w) - \frac{\mathbf{1} - \boldsymbol{\theta}}{\boldsymbol{\theta}} \ln(\mathbf{J}) \right) = cst$$

Noting:

$$\log A_s \approx \alpha_s \left(-\mu_\tau + \frac{1}{2} \frac{\theta_s}{1 - \theta_s} \left(\alpha \sigma_\tau^2 - 2\sigma_{z\tau} \right) \right)$$

we obtain:

$$\ln w = \frac{\boldsymbol{\phi'} \left(I - \Gamma'\right)^{-1} \ln(\mathbf{A})}{\boldsymbol{\phi'} \left(I - \Gamma'\right)^{-1} \boldsymbol{\beta}} + cst$$

which leads to the equation in the proposition.

Formula for TFP

Let us first define the right exponents for TFP. We want these exponents to satisfy the property that, in the absence of distortions and markups, aggregate output satisfies $Y = AK^{\alpha^*} L^{1-\alpha^*}$ and that this aggregate production function defines the same price system as the decentralized economy. Hence, α^* needs to satisfy that:

$$\alpha^* = \frac{RK}{Y} \text{ and } 1 - \alpha^* = \frac{wL}{Y}$$

when there is *no distortion (mark-ups nor wedges)*.

Remember that at the level of industry s labor demand is given by:

$$wL = \sum_{s=1}^{S} \theta_s \beta_s P_s Q_s$$

Simultaneously, we know that industry level market clearing condition can be written as (in vector terms):

$$\boldsymbol{PQ} = Y (I - \Psi)^{-1} \boldsymbol{\phi}$$

where $\Psi = \Theta\Gamma$ is the input-output matrix of revenue shares ($\theta_s \gamma_{js}$). We have that $\Psi = \Gamma$ if and only if θ_s, i.e. there is no mark-up. We assume this in the benchmark situaton.

Assuming perfect competition ($\theta = 1$), and combining these two equations, we obtain that:

$$wL = Y \boldsymbol{\beta}' (I - \Gamma)^{-1} \boldsymbol{\phi}$$

So we set:

$$\boxed{\alpha^* = 1 - \boldsymbol{\beta}' (I - \Gamma)^{-1} \boldsymbol{\phi}}$$

Now, we show that the power on K is indeed α^*. We start from the FOC in K, assume no distiortion, and aggregate up to the sector, then economy level, we find:

$$RK = \sum_s \alpha_s \theta_s P_s Q_s$$

which leads to:

$$RK = Y \boldsymbol{\alpha}' (I - \Gamma)^{-1} \boldsymbol{\phi}$$

We use the fact that $\boldsymbol{\alpha} + \boldsymbol{\beta} = (I - \Gamma')\mathbf{1}$ to show that the sum of labor and capital shares equal to 1 (in the no mark-up, no distortion case).

So we finally arrive at the result that $Y = AK^{\alpha^*} L^{1-\alpha^*}$ is a production function endowed with the right properties when there are no distortions.

Hence, we compute $TFP = \frac{Y}{K^{\alpha^*} L^{1-\alpha^*}}$. This measures the production loss compared to the unconstrained, undistorted case.

We use the fact that:

$$\log TFP = -\alpha^* \log \frac{K}{Y} - (1 - \alpha^*) \log \frac{L}{Y}$$

We start with the labor term. Knowing that $wL_s = \alpha_s \theta_s P_s Q_s$ and that $P_s Q_s$ are fixed fractions of Y, we obtain that $\frac{L}{Y}$ is proportional fo $\frac{1}{w}$, hence, given our final derivation for log output:

$$\log \frac{L}{Y} = -\frac{\boldsymbol{\phi'} \left(I - \Gamma'\right)^{-1} \ln(\mathbf{A})}{\boldsymbol{\phi'} \left(I - \Gamma'\right)^{-1} \boldsymbol{\beta}} + cst$$

We note ϕ_s^* the sth element of $\left(I - \Gamma\right)^{-1} \boldsymbol{\phi}$, as the linkage-adjusted industry share. Then:

$$\Delta \log \frac{L}{Y} = -\sum_s \left(\frac{\alpha_s \phi_s^*}{\sum_{s'} \beta_s \phi_{s'}^*} \right) \left(-\Delta \mu_\tau(s) + \frac{1}{2} \frac{\theta_s}{1 - \theta_s} \left(\alpha_s \Delta \sigma_\tau^2(s) - 2\Delta \sigma_{z\tau}(s) \right) \right)$$

Capital productivity is a bit trickier. Start with the fact that:

$$\frac{Y}{K} = \sum_s \frac{K_s}{K} \frac{Y_s}{K_s}$$

We need to calculate K_s. Note that:

$$p_{is} y_{is} = \frac{P_s Q_s}{J_s} \frac{e^{\frac{\theta_s}{1-\theta_s} z_i}}{(1 + \tau_i)^{\frac{\alpha_s \theta_s}{1-\theta_s}}}$$

so that capital demand is given by:

$$k_i s = \theta_s \alpha_s \frac{P_s Q_s}{R J_s} \frac{e^{\frac{\theta_s}{1-\theta_s} z_i}}{(1 + \tau_i)^{1+\frac{\alpha_s \theta_s}{1-\theta_s}}}$$

so that the industry level capital stock is:

$$K_s = \theta_s \alpha_s \frac{P_s Q_s}{R J_s} \underbrace{\int \frac{e^{\frac{\theta_s}{1-\theta_s} z_i}}{(1+\tau_i)^{1+\frac{\alpha_s \theta_s}{1-\theta_s}}} di}_{I_s}$$

Using the fact that $\boldsymbol{PQ} = (I - \Psi)^{-1} \phi Y$, we obtain

$$\frac{K_s}{Y} = \eta_s \frac{I_s}{J_s}$$

where η_s is a function of parameters. Hence:

$$\log \frac{K}{Y} = \log \left(\sum_s \eta_s \frac{I_s}{J_s} \right)$$

We log-linearize this expression by setting $\delta_s = \Delta \log \frac{I_s}{J_s}$

$$\begin{aligned}
\log \frac{K^1}{Y^1} &= \log \left(\sum_s \eta_s \frac{I_s^0}{J_s^0} e^{\delta_s} \right) \\
&\approx \log \left(\sum_s \eta_s \frac{I_s^0}{J_s^0} \left(1 + \delta_s + \frac{\delta_s^2}{2} \right) \right) \\
&\approx \log \frac{K^0}{Y^0} + \log \left(1 + \sum_s \kappa_s \left(\delta_s + \frac{\delta_s^2}{2} \right) \right) \\
&\approx \log \frac{K^0}{Y^0} + \sum_s \kappa_s \delta_s + \frac{1}{2} \left(\sum_s \kappa_s \frac{\delta_s^2}{2} - \left(\sum_s \kappa_s \delta_s \right)^2 \right)
\end{aligned}$$

where $\kappa_s = \frac{K_s^0}{K^0}$ is the capital share of each industry.

A second order Taylor expansion leads to:

$$\delta_s = -\Delta \mu_\tau(s) + \frac{1}{2} \left(\left(1 + 2 \frac{\alpha_s \theta_s}{1-\theta_s} \right) \Delta \sigma_\tau^2(s) - 2 \frac{\theta_s}{1-\theta_s} \Delta \sigma_{z\tau}(s) \right)$$

Keeping only second order terms, we obtain

We combine the expressions for labor and capital productivities and finally obtain:

$$\Delta \log \frac{K}{Y} = \sum_s \kappa_s \left(-\Delta \mu(s) + \frac{1}{2} \left(\left(1 + 2 \frac{\alpha_s \theta_s}{1-\theta_s} \right) \Delta \sigma_\tau^2(s) - 2 \frac{\theta_s}{1-\theta_s} \Delta \sigma_{z\tau}(s) \right) \right) + \frac{1}{2} \Delta \mathbf{var}_{\kappa_s} \left(\mu(s) \right)$$

This leads to the TFP formula:

$$\begin{aligned}
\Delta \log TFP = &-\alpha^* \sum_s \kappa_s \left(-\Delta \mu(s) + \frac{1}{2} \left(\left(1 + 2 \frac{\alpha_s \theta_s}{1-\theta_s} \right) \Delta \sigma_\tau^2(s) - 2 \frac{\theta_s}{1-\theta_s} \Delta \sigma_{z\tau}(s) \right) \right) - \frac{\alpha^*}{2} \Delta \mathbf{var}_{\kappa_s} \left(\mu(s) \right) \\
&+ (1-\alpha^*) \sum_s \left(\frac{\alpha_s \phi_s^*}{\sum_{s'} \beta_{s'} \phi_{s'}^*} \right) \left(-\Delta \mu_\tau(s) + \frac{1}{2} \frac{\theta_s}{1-\theta_s} \left(\alpha_s \Delta \sigma_\tau^2(s) - 2 \Delta \sigma_{z\tau}(s) \right) \right)
\end{aligned}$$

Using the fact that $1-\alpha^* = \sum_s \beta_s \phi_s^*$, we obtain that:

$$\Delta \log TFP = -\alpha^* \sum_s \kappa_s \left(-\Delta\mu(s) + \frac{1}{2}\left(\left(1+2\frac{\alpha_s\theta_s}{1-\theta_s}\right)\Delta\sigma_\tau^2(s) - 2\frac{\theta_s}{1-\theta_s}\Delta\sigma_{z\tau}(s)\right)\right) - \frac{\alpha^*}{2}\Delta\text{var}_{\kappa_s}(\mu(s))$$
$$+\sum_s \alpha_s\phi_s^* \left(-\Delta\mu_\tau(s) + \frac{1}{2}\frac{\theta_s}{1-\theta_s}\left(\alpha_s\Delta\sigma_\tau^2(s) - 2\Delta\sigma_{z\tau}(s)\right)\right)$$

A.7 Proof of Proposition 6

We consider here S heterogeneous industries with M_s firms operating in industry s. The setup is similar to section 5 except that there is no input-output linkages and that the final good market produces by combining industry outputs according to a CES production function:

$$Y = \left(\sum_{k=1}^{S} \chi_s Y_s^\psi\right)^{\frac{1}{\psi}}, \text{ with } \sum_{s=1}^{S}\chi_s = 1$$

Each firm within industry s has a Cobb-Douglas production function with the same factor shares, however the price-elasticity of demand is allowed to vary by industry

$$y_{it} = e^{z_{it}} k_{it}^{\alpha} l_{it}^{1-\alpha}, \qquad Y_s = \left(\int_i y_{is}^{\theta_s}\right)^{\frac{1}{\theta_s}}$$

Profit maximization gives the demand for industry s and firm i output:

$$\max_{Y_s} PY - \sum_{s=1}^{S} P_s Y_s = \max_{Y_s} P\left(\sum_{k=1}^{S}\chi_s Y_s^\psi\right)^{\frac{1}{\psi}} - \sum_{s=1}^{S} P_s Y_s$$

$$\text{FOC w.r.t } Y_s : P\chi_s Y_s^{\psi-1}\left(\sum_{k=1}^{S}\chi_s Y_s^\psi\right)^{\frac{1}{\psi}-1} = P_s$$

$$\implies \frac{P_s}{P} = \chi_s\left(\frac{Y_s}{Y}\right)^{\psi-1}, \text{ and similarly } \frac{p_{is}}{P_s} = \left(\frac{y_{is}}{Y_s}\right)^{\theta_s-1} \tag{14}$$

labor demand by each firm is given by the following problem:

$$\max_{l_{is}}(p_{is}y_{is} - wl_{is}) = \max_{l_{is}}\left(\chi_s P\left(\frac{Y_s}{Y}\right)^{\psi-1} Y_s^{1-\theta_s}\right) y_{is}^{\theta_s} - wl_{is}$$

$$\implies l_{is} = \left(\frac{(1-\alpha)\theta_s}{w}\right)^{\frac{1}{1-(1-\alpha)\theta_s}} \left(\chi_s P\left(\frac{Y_s}{Y}\right)^{\psi-1} Y_s^{1-\theta_s}\right)^{\frac{1}{1-(1-\alpha)\theta_s}} e^{\frac{\theta_s}{1-(1-\alpha)\theta_s} z_{is}} k_{is}^{\frac{\alpha\theta_s}{1-(1-\alpha)\theta_s}}$$

we have, for each firm in industry s: $(1-\alpha)\theta_s p_{is} y_{is} = w l_{is}$. Replacing above yields:

$$p_{is} y_{is} = \left(\frac{(1-\alpha)\theta_s}{w}\right)^{\frac{(1-\alpha)\theta_s}{1-(1-\alpha)\theta_s}} \left(\chi_s P \left(\frac{Y_s}{Y}\right)^{\psi-1} Y_s^{1-\theta_s}\right)^{\frac{1}{1-(1-\alpha)\theta_s}} e^{\frac{\theta_s}{1-(1-\alpha)\theta_s} z_{is}} k_{is}^{\frac{\alpha\theta_s}{1-(1-\alpha)\theta_s}} \tag{15}$$

Letting $(1+\tau_{is})R = \alpha\theta_s \frac{p_{is} y_{is}}{k_{is}}$ in the above equation we get

$$k_{is} = \left(\chi_s \left(\frac{Y}{Y_s}\right)^{1-\psi} Y_s^{1-\theta_s}\right)^{\frac{1}{1-\theta_s}} \left(\frac{\alpha\theta_s}{(1+\tau_{is})R}\right)^{\frac{1-(1-\alpha)\theta_s}{1-\theta_s}} \left(\frac{(1-\alpha)\theta_s}{w}\right)^{\frac{(1-\alpha)\theta_s}{1-\theta_s}} e^{\frac{\theta_s}{1-\theta_s} z_{is}}$$

$$\implies p_{is} y_{is} = \left(\chi_s \left(\frac{Y}{Y_s}\right)^{1-\psi} Y_s^{1-\theta_s}\right)^{\frac{1}{1-\theta_s}} \left(\frac{\alpha\theta_s}{R}\right)^{\frac{\alpha\theta_s}{1-\theta_s}} \left(\frac{(1-\alpha)\theta_s}{w}\right)^{\frac{(1-\alpha)\theta_s}{1-\theta_s}} \left(\frac{e^{z_{is}}}{(1+\tau_{is})^{\alpha}}\right)^{\frac{\theta_s}{1-\theta_s}}$$

using equation (14) we get the firm level demand function:

$$y_{is} = \left(\chi_s \left(\frac{Y}{Y_s}\right)^{1-\psi} Y_s^{1-\theta_s}\right)^{\frac{1}{1-\theta_s}} \left(\frac{\alpha\theta_s}{R}\right)^{\frac{\alpha}{1-\theta_s}} \left(\frac{(1-\alpha)\theta_s}{w}\right)^{\frac{(1-\alpha)}{1-\theta_s}} \left(\frac{e^{z_{is}}}{(1+\tau_{is})^{\alpha}}\right)^{\frac{1}{1-\theta_s}}$$

Aggregating within the industry, and since $Y_s = (\int_i y_{is}^{\theta_s})^{\frac{1}{\theta_s}}$:

$$\left(\frac{Y_s}{Y}\right)^{(1-\psi)\frac{\theta_s}{1-\theta_s}} = \chi_s^{\frac{\theta_s}{1-\theta_s}} \left(\frac{\alpha\theta_s}{R}\right)^{\frac{\alpha\theta_s}{1-\theta_s}} \left(\frac{(1-\alpha)\theta_s}{w}\right)^{\frac{(1-\alpha)\theta_s}{1-\theta_s}} \underbrace{\left(\int_i \left(\frac{e^{z_{is}}}{(1+\tau_{is})^{\alpha}}\right)^{\frac{\theta_s}{1-\theta_s}} di\right)}_{I_s}$$

Summing across industries pins down the wage. Since $Y = \left(\sum_{s=1}^{S} \chi_s Y_s^{\psi}\right)^{\frac{1}{\psi}}$, we have :

$$1 = \sum_{s=1}^{S} \chi_s^{\frac{1}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\frac{\alpha\psi}{1-\psi}} \left(\frac{(1-\alpha)\theta_s}{w}\right)^{\frac{(1-\alpha)\psi}{1-\psi}} I_s^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}$$

$$\implies w = \left[\sum_{s=1}^{S} \chi_s^{\frac{1}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\frac{\alpha\psi}{1-\psi}} ((1-\alpha)\theta_s)^{\frac{(1-\alpha)\psi}{1-\psi}} I_s^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}\right]^{\frac{1-\psi}{(1-\alpha)\psi}}$$

Aggregate labor market clearing yields the following:

$$(1-\alpha)\theta_s p_{is} y_{is} = w l_{is} \implies L_s = \frac{(1-\alpha)\theta_s}{w} P_s Y_s$$

$$\bar{L}\left(\frac{w}{\bar{w}}\right)^{1+\epsilon} = \sum_{s=1}^{S} \left(\frac{(1-\alpha)\theta_s}{\bar{w}}\right) P_s Y_s = Y \sum_{s=1}^{S} \left(\frac{(1-\alpha)\theta_s}{\bar{w}}\right) \chi_s \left(\frac{Y_s}{Y}\right)^{\psi}$$

Therefore:

$$Y = \frac{\frac{\bar{L}}{\bar{w}^{1+\epsilon}} \left[\sum_{s=1}^{S} \chi_s^{\frac{1}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\frac{\alpha\psi}{1-\psi}} ((1-\alpha)\theta_s)^{\frac{(1-\alpha)\psi}{1-\psi}} I_s^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}\right]^{1+\frac{(1-\psi)(1+\epsilon)}{(1-\alpha)\psi}}}{\sum_{s=1}^{S} \left(\frac{(1-\alpha)\theta_s}{\bar{w}}\right) \chi_s^{\frac{1}{1-\psi}} ((1-\alpha)\theta_s)^{(1-\alpha)\frac{\psi}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\alpha\frac{\psi}{1-\psi}} I_s^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}}$$

Let $I_s(1)$ (resp. $I_s(0)$) be the value of I_s when $\Theta = \Theta_1$ (resp. Θ_0). We start with the effect of treatment on the log numerator:

$$\Delta_1 = \ln\left[\sum_{s=1}^{S} \chi_s^{\frac{1}{1-\psi}} ((1-\alpha)\theta_s)^{(1-\alpha)\frac{\psi}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\alpha\frac{\psi}{1-\psi}} I_s(1)^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}\right]$$

$$-\ln\left[\sum_{s=1}^{S} \chi_s^{\frac{1}{1-\psi}} ((1-\alpha)\theta_s)^{(1-\alpha)\frac{\psi}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\alpha\frac{\psi}{1-\psi}} I_s(0)^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}\right]$$

$$= \ln\left(\frac{\sum_{s=1}^{S} \chi_s^{\frac{1}{1-\psi}} \left(\frac{(1-\alpha)\theta_s}{w_0}\right)^{(1-\alpha)\frac{\psi}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\alpha\frac{\psi}{1-\psi}} I_s(1)^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}}{\sum_{s=1}^{S} \chi_s^{\frac{1}{1-\psi}} \left(\frac{(1-\alpha)\theta_s}{w_0}\right)^{(1-\alpha)\frac{\psi}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\alpha\frac{\psi}{1-\psi}} I_s(0)^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}}\right)$$

$$= \ln\left(\frac{\sum_{s=1}^{S} \chi_s^{\frac{1}{1-\psi}} \left(\frac{(1-\alpha)\theta_s}{w_0}\right)^{(1-\alpha)\frac{\psi}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\alpha\frac{\psi}{1-\psi}} I_s(0)^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}} e^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}\Delta_s}}{\sum_{s=1}^{S} \chi_s^{\frac{1}{1-\psi}} \left(\frac{(1-\alpha)\theta_s}{w_0}\right)^{(1-\alpha)\frac{\psi}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\alpha\frac{\psi}{1-\psi}} I_s(0)^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}}\right)$$

Recall that

$$\chi_s \left(\frac{Y_s^0}{Y^0}\right)^{\psi} = \frac{P_s^0 Y_s^0}{P^0 Y^0} = \chi_s^{\frac{1}{1-\psi}} \left(\frac{(1-\alpha)\theta_s}{w_0}\right)^{(1-\alpha)\frac{\psi}{1-\psi}} \left(\frac{\alpha\theta_s}{R}\right)^{\alpha\frac{\psi}{1-\psi}} I_s(0)^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}$$

$$\implies \Delta_1 = \ln\left(\frac{\sum_{s=1}^{S} \frac{P_s^0 Y_s^0}{P^0 Y^0} e^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}\Delta_s}}{\sum_{s=1}^{S} \frac{P_s^0 Y_s^0}{P^0 Y^0}}\right) = \ln\left(\sum_{s=1}^{S} \frac{P_s^0 Y_s^0}{P^0 Y^0} e^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}\Delta_s}\right)$$

The treatment effect on each industry is weighted by its share of the total output in the initial economy. Let $\gamma_s = \frac{P_s^0 Y_s^0}{P^0 Y^0}$, and $u_s = \frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}\Delta_s$. The Taylor expansion gives:

$$\Delta_1 = \ln\left(\sum_{s=1}^{S} \gamma_s e^{u_s}\right) \approx \ln\left(\sum_{s=1}^{S} \gamma_s \left(1 + u_s + \frac{u_s^2}{2}\right)\right)$$

$$\Delta_1 = \ln\left(1 + \sum_{s=1}^{S} \gamma_s u_s + \frac{1}{2}\sum_{s=1}^{S} \gamma_s u_s^2\right) \approx \sum_{s=1}^{S} \gamma_s u_s + \frac{1}{2}\sum_{s=1}^{S} \gamma_s u_s^2 - \frac{1}{2}\left(\sum_{s=1}^{S} \gamma_s u_s\right)^2$$

Replace u_s with its approximation value:

$$u_s \approx \frac{\psi}{1-\psi}\left[-\alpha\Delta\mu_\tau(s) + \frac{1}{2}\left(\frac{\theta_s}{1-\theta_s}\right)\left(-2\alpha\Delta\sigma_{z\tau}(s) + \alpha^2\Delta\sigma_\tau^2(s)\right)\right] \tag{16}$$

$$\implies \Delta_1 \approx \sum_{s=1}^{S} \frac{\gamma_s\alpha\psi}{1-\psi}\left[-\Delta\mu_\tau(s) + \frac{1}{2}\left(\frac{\theta_s}{1-\theta_s}\right)\left(-2\Delta\sigma_{z\tau}(s) + \alpha\Delta\sigma_\tau^2(s)\right)\right]$$

$$+\frac{1}{2}\left(\frac{\psi}{1-\psi}\right)^2\sum_{s=1}^{S}\gamma_s\alpha^2\left(\Delta\mu_\tau(s)\right)^2-\frac{1}{2}\left(\frac{\psi}{1-\psi}\right)^2\left(\sum_{s=1}^{S}\gamma_s\alpha\Delta\mu_\tau(s)\right)^2$$

For the effect of treatment on log denominator (Δ_2), we have:

$$\Delta_2=\ln\left(\sum_{s=1}^{S}\left(\frac{(1-\alpha)\theta_s}{\bar{w}}\right)\chi_s^{\frac{1}{1-\psi}}\left((1-\alpha)\theta_s\right)^{\frac{(1-\alpha)\psi}{1-\psi}}\left(\frac{\alpha\theta_s}{R}\right)^{\frac{\alpha\psi}{1-\psi}}I_s(1)^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}\right)$$

$$-\ln\left(\sum_{s=1}^{S}\left(\frac{(1-\alpha)\theta_s}{\bar{w}}\right)\chi_s^{\frac{1}{1-\psi}}\left((1-\alpha)\theta_s\right)^{\frac{(1-\alpha)\psi}{1-\psi}}\left(\frac{\alpha\theta_s}{R}\right)^{\frac{\alpha\psi}{1-\psi}}I_s(0)^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}}\right)$$

$$\Delta_2=\ln\left(\frac{\sum_{s=1}^{S}\theta_s\frac{P_s^0Y_s^0}{P^0Y^0}e^{\frac{\psi}{1-\psi}\frac{1-\theta_s}{\theta_s}\Delta_s}}{\sum_{s=1}^{S}\theta_s\frac{P_s^0Y_s^0}{P^0Y^0}}\right)=\ln\left(\sum_{s=1}^{S}\theta_s\gamma_s e^{u_s}\right)-\ln\left(\sum_{s=1}^{S}\theta_s\gamma_s\right)$$

$$\approx\ln\left(\sum_{s=1}^{S}\theta_s\gamma_s\left(1+u_s+\frac{u_s^2}{2}\right)\right)-\ln\left(\sum_{s=1}^{S}\theta_s\gamma_s\right)=\ln\left(1+\frac{\sum_{s=1}^{S}\theta_s\gamma_s u_s+\frac{1}{2}\sum_{s=1}^{S}\theta_s\gamma_s u_s^2}{\sum_{s=1}^{S}\theta_s\gamma_s}\right)$$

$$\Delta_2\approx\frac{\sum_{s=1}^{S}\theta_s\gamma_s u_s+\frac{1}{2}\sum_{s=1}^{S}\theta_s\gamma_s u_s^2}{\sum_{s=1}^{S}\theta_s\gamma_s}-\frac{1}{2}\left(\frac{\sum_{s=1}^{S}\theta_s\gamma_s u_s}{\sum_{s=1}^{S}\theta_s\gamma_s}\right)^2$$

Replacing u_s with equation (16) gives:

$$\Delta_2\approx\frac{\psi}{(1-\psi)}\sum_{s=1}^{S}\alpha w_s\left[-\Delta\mu_\tau(s)+\frac{1}{2}\left(\frac{\theta_s}{1-\theta_s}\right)\left(-2\Delta\sigma_{z\tau}(s)+\alpha\Delta\sigma_\tau^2(s)\right)\right]$$

$$+\frac{1}{2}\left(\frac{\psi\alpha}{1-\psi}\right)^2\sum_{s=1}^{S}w_s\left(\Delta\mu_\tau(s)\right)^2-\frac{1}{2}\left(\frac{\psi\alpha}{1-\psi}\right)^2\left(\sum_{s=1}^{S}w_s\Delta\mu_\tau(s)\right)^2$$

after noting $w_s=\frac{\theta_s\gamma_s}{\sum_{s=1}^{S}\theta_s\gamma_s}$.

The overall effect of the reform on aggregate output is then given by:

$$\Delta\ln(Y)=\left(1+\frac{(1-\psi)(1+\epsilon)}{(1-\alpha)\psi}\right)\Delta_1-\Delta_2$$

We inject the two expressions for Δ_1 and Δ_2 and obtain:

$$\begin{aligned}\Delta\ln(Y)=&\frac{(1+\epsilon)\alpha}{1-\alpha}\sum_{s=1}^{S}\gamma_s\left[-\Delta\mu_\tau(s)+\frac{1}{2}\left(\frac{\theta_s}{1-\theta_s}\right)\left(-2\Delta\sigma_{z\tau}(s)+\alpha\Delta\sigma_\tau^2(s)\right)\right]\\&+\frac{1}{2}\left(\frac{\psi\alpha}{1-\psi}\right)^2\left[(1+\epsilon)\frac{1-\psi}{\psi(1-\alpha)}Var^\star_{\gamma_s}\left(\Delta\mu_\tau(s)\right)\right]\\&+\frac{\alpha\psi}{1-\psi}\sum_{s=1}^{S}(\gamma_s-w_s)\left[-\Delta\mu_\tau(s)+\frac{1}{2}\left(\frac{\theta_s}{1-\theta_s}\right)\left(-2\Delta\sigma_{z\tau}(s)+\alpha\Delta\sigma_\tau^2(s)\right)\right]\\&+\frac{1}{2}\left(\frac{\psi\alpha}{1-\psi}\right)^2\left[Var^\star_{\gamma_s}\left(\Delta\mu_\tau(s)\right)-Var^\star_{w_s}\left(\Delta\mu_\tau(s)\right)\right]\end{aligned}$$

where $Var_z^\star(\Delta\mu_\tau(s))$ is the cross-sectional variance of treatments across industries, weighed by z.

A.8 Proof of Proposition 7

In this Appendix, we derive approximate aggregation formulas in the case where (1) intermediate input producers follow a CES technology as in Equation (12) and (2) production is organized as in Section 2.

A.8.1 Equilibrium Formulas & Notations

Given the firm-level production function, firm output is given by: $p_i y_i = Y^{1-\theta} e^{\theta z_i}[\alpha k_i^\rho + (1-\alpha) l_i^\rho]^{\frac{\theta}{\rho}}$. We combine this equation with the definition of the capital wedge τ_i and the static FOC in labor. We obtain:

$$\begin{cases} k_i = Y e^{\frac{\theta}{1-\theta} z_i} \left(\dfrac{\alpha\theta}{R(1+\tau_i)}\right)^{\frac{1}{1-\rho}} \left[\alpha \left(\dfrac{\alpha\theta}{R(1+\tau_i)}\right)^{\frac{\rho}{1-\rho}} + (1-\alpha)\left(\dfrac{(1-\alpha)\,\theta}{w}\right)^{\frac{\rho}{1-\rho}}\right]^{\frac{\theta-\rho}{\rho(1-\theta)}} \\ l_i = Y e^{\frac{\theta}{1-\theta} z_i} \left(\dfrac{(1-\alpha)\theta}{w}\right)^{\frac{1}{1-\rho}} \left[\alpha \left(\dfrac{\alpha\theta}{R(1+\tau_i)}\right)^{\frac{\rho}{1-\rho}} + (1-\alpha)\left(\dfrac{(1-\alpha)\,\theta}{w}\right)^{\frac{\rho}{1-\rho}}\right]^{\frac{\theta-\rho}{\rho(1-\theta)}} \end{cases},$$

and optimal firm-level output is simply:

$$p_i y_i = e^{z_i \frac{\theta}{1-\theta}} Y \left[\alpha \left(\frac{\alpha\theta}{R(1+\tau_i)}\right)^{\frac{\rho}{1-\rho}} + (1-\alpha)\left(\frac{(1-\alpha)\,\theta}{w}\right)^{\frac{\rho}{1-\rho}}\right]^{\frac{1-\rho}{\rho}\frac{\theta}{1-\theta}},$$

Note μ and σ_τ^2 the mean and variance of $\log(1+\tau)$. Furthermore, note:

$$\begin{aligned} a &= (1-\alpha)\left(\frac{(1-\alpha)\,\theta}{w^*}\right)^{\frac{\rho}{1-\rho}} \\ b &= \alpha\left(\frac{\alpha\theta}{Re^\mu}\right)^{\frac{\rho}{1-\rho}} \end{aligned}$$

where w^* is the wage prevailing in a fictitious economy where $\log(1+\tau_i) = \mu$ for all firms. We note $\delta_w = \log(w/w^*)$ and $\delta_i = \log(1+\tau_i) - \mu$ the deviation of the equilibrium from this hypothetical equilibrium with homogeneous distiortion.

The general equilibrium of this economy (w,Y) is defined by the labor and product market clearing conditions:

$$\begin{cases} 1 = \displaystyle\int_{z,\tau} e^{z_i \frac{\theta}{1-\theta}} \left[a e^{-\frac{\rho}{1-\rho}\delta_w} + b e^{-\frac{\rho}{1-\rho}\delta_i}\right]^{\frac{1-\rho}{\rho}\frac{\theta}{1-\theta}} di \\ \bar{L}\left(\dfrac{w}{\bar{w}}\right)^\epsilon = Y\left(\dfrac{(1-\alpha)\theta}{w}\right)^{\frac{1}{1-\rho}} \displaystyle\int_i e^{\frac{\theta z_i}{1-\theta}} \left[a e^{-\frac{\rho}{1-\rho}\delta_w} + b e^{-\frac{\rho}{1-\rho}\delta_i}\right]^{\frac{\theta-\rho}{\rho(1-\theta)}} di \end{cases}$$

which define w and Y as functions of the moments of the distributions of log distortions and log productivity.

Note that, by definition of a, b, we have that:

$$
\begin{cases}
1 = (a+b)^{\frac{1-\rho}{\rho}\frac{\theta}{1-\theta}} \displaystyle\int_{z,\tau} e^{z_i \frac{\theta}{1-\theta}} di \\
\bar{L}\left(\dfrac{w^*}{\bar{w}}\right)^{\epsilon} = Y^* \, (a+b)^{\frac{\theta-\rho}{\rho(1-\theta)}} \left(\dfrac{(1-\alpha)\theta}{w^*}\right)^{\frac{1}{1-\rho}} \displaystyle\int_i e^{\frac{\theta z_i}{1-\theta}} di
\end{cases}
$$

A.8.2 Linearization

We want to compute the change in log output due to the aggregation of the reform. Index with 0 the moments of $\log(1+\tau_i)$ without the policy, and with 1 the moments of log distortions when the policy is applied to all firms. Then, we decompose the change in log output into:

$$
\begin{aligned}
\Delta logY &= \log Y(\{\log(1+\tau_i^1)\}_i) - \log Y(\{\log(1+\tau_i^0)\}_i) \\
&= \underbrace{\log Y(\{\log(1+\tau_i^1)\}_i) - \log Y(\mu_1)}_{H_1^Y} \\
&+ \underbrace{\log Y(\mu_1) - \log Y(\mu_0)}_{M^Y} \\
&- \underbrace{\left(\log Y(\mu_0) - \log Y(\{\log(1+\tau_i^0)\}_i)\right)}_{H_0^Y}
\end{aligned}
$$

where H_i^Y is the pure effect of heterogeneous distortions, starting from an economy where all log wedges are equal to μ_i. M^Y is the pure effect of changing the mean distiortion, assuming no heterogeneity. We use the same decomposition to compute $\Delta \log w$.

In the following, we make two separate assumptions. First, we assume that in both treated and untreated economies, that log productivity and log wedges experience small deviation from their means. This allows to compute H_0^Y and H_1^Y. Second, we assume that the experiment has a small impact on the mean distiortion, i.e. that $\Delta\mu = \mu_1 - \mu_0$ is small. This second assumption allows us to compute M^Y. We use these two assumptions to expand the equlibrium formulas up to the second order.

First, we note that a second order Taylor expansion in z, combined with the product market market equilibrium leads to:

$$
a + b = 1 - \frac{1}{2}\frac{\rho}{1-\rho}\frac{\theta}{1-\theta}\sigma_z^2
$$

so that terms in $a+b$ multiplied by first- or second-order terms are equal to 1.

The pure effect on the mean wedge is given by:

$$
\begin{aligned}
M^w &= -\frac{b}{a}\Delta\mu + \frac{b^2}{2a}\frac{\rho}{1-\rho}(\Delta\mu)^2 \\
M^Y &= \left(\epsilon + \frac{1}{1-\rho}\right)\left(-\frac{b}{a}\Delta\mu + \frac{b^2}{2a}\frac{\rho}{1-\rho}(\Delta\mu)^2\right)
\end{aligned}
$$

The pure effect of wedge heterogeneity is given by (omitting the $0,1$ index):

$$
\begin{aligned}
H^w &= \frac{b}{2a}\left(\frac{\rho}{1-\rho}a + \frac{\theta}{1-\theta}b\right)\sigma_\tau^2 - \frac{b}{a}\frac{\theta}{1-\theta}\sigma_{z\tau} \\
H^Y &= \frac{1}{2(1-\theta)}\left(b^2\frac{\rho(\theta-\rho)}{(1-\rho)^2}a + \left(\epsilon + \frac{1}{1-\rho}\right)\left(\frac{\rho-\theta}{1-\theta}b + \theta\frac{b}{a}\right)\right)\sigma_\tau^2 - \left(\epsilon + \frac{1}{1-\rho}\right)\frac{b}{a}\frac{\theta}{1-\theta}\sigma_{z\tau}
\end{aligned}
$$

where a and b are given by the above formulas. In particular, a depends on μ, the expectation of log wedges and b depends on w^*, the wage prevailing in an economy where the average log wedge is μ but the dispersion is zero.

To compute the change in output (the same applies to wage), we need to compute $H_1^Y - H_0^Y$, which involves two sets of coefficients (a_1, b_1) (which depend on μ_1 and w_1^*)and (a_0, b_0) (which depend on μ_0 and w_0^*).

We can write that:

$$
\begin{aligned}
a_1 &= a_0\left(1 + \frac{\rho}{1-\rho}\frac{b_0}{a_0}\Delta\mu + O(2)\right) \\
b_1 &= b_0\left(1 - \frac{\rho}{1-\rho}\Delta\mu + O(2)\right)
\end{aligned}
$$

which shows that the difference between the two coefficients is of order one. Given that H^w and H^Y multiply these coefficients by terms of order 2, we obtain directly that $H_1^w = H_0^w$ and $H_1^Y = H_0^Y$.

Hence, noting $a = a_0$ and $b = b_0$, the formula for $\Delta \log Y$ can be shown to be:

$$
\begin{aligned}
\Delta \log Y = &-\frac{b}{a}\left(\epsilon + \frac{1}{1-\rho}\right)\Delta\mu \\
&+ \frac{1}{2(1-\theta)}\left(b^2\frac{\rho(\theta-\rho)}{(1-\rho)^2} + \left(\epsilon + \frac{1}{1-\rho}\right)\left(\frac{\rho-\theta}{1-\rho}b + \theta\frac{b}{a}\right)\right)\Delta\sigma_\tau^2 \\
&- \left(\epsilon + \frac{1}{1-\rho}\right)\frac{b}{a}\frac{\theta}{1-\theta}\Delta\sigma_{z\tau} \\
&+ \left(\epsilon + \frac{1}{1-\rho}\right)\frac{b}{2a^2}\frac{\rho}{1-\rho}(\Delta\mu)^2
\end{aligned}
$$

while the formula for $\Delta \log w$ is:

$$
\begin{aligned}
\Delta \log w = &-\frac{b}{a}\Delta\mu \\
&+ \frac{b}{2a}\left(\frac{\rho}{1-\rho}a + \frac{\theta}{1-\rho}b\right)\Delta\sigma_\tau^2 \\
&- \frac{b}{a}\frac{\theta}{1-\theta}\Delta\sigma_{z\tau} \\
&+ \frac{b}{2a^2}\frac{\rho}{1-\rho}(\Delta\mu)^2
\end{aligned}
$$

with:

$$a = (1-\alpha)\left(\frac{(1-\alpha)\,\theta}{w_0^*}\right)^{\frac{\rho}{1-\rho}}$$
$$b = \alpha\left(\frac{\alpha\theta}{Re^{\mu_0}}\right)^{\frac{\rho}{1-\rho}}$$

A.8.3 Measurement

In order to use the above formulas, two challenges present themselves. First, we need to retrieve a and b from the data. Second, we need to obtain $\Delta\mu$ and $\Delta\sigma_\tau$. Let us tackle these two problems in turn.

Measuring a and b

To retrieve a and b, we start from the formula of the labor share:

$$s_i^L = \theta\frac{ae^{-\frac{\rho}{1-\rho}\delta_w}}{ae^{-\frac{\rho}{1-\rho}\delta_w} + be^{-\frac{\rho}{1-\rho}\delta_i}}$$

We expand this formula to the second order and take the expectation of s_i^L, which can be directly measured in the data:

$$a = \frac{\widehat{s_L}}{\theta}\left(1 + \left(\frac{1}{2} + a\right)b\left(\frac{\rho}{1-\rho}\right)^2\sigma_\tau^2\right)$$

where $\widehat{s_L}$ is the average labor share. Hence, $a = \frac{\widehat{s_L}}{\theta} + O(2)$. Given that a is multiplied by terms of order 1 or 2, the difference between a and $\frac{\widehat{s_L}}{\theta}$ is negligible at the second order of approximation.

We can then use the product equilibrium condition to see that:

$$a + b = 1 - \frac{1}{2}\frac{\theta}{1-\theta}\frac{\rho}{1-\rho}\sigma_z^2$$

hence, for the same reason as above, we can approximate b with $1 - \frac{\widehat{s_L}}{\theta}$.

Measuring $\Delta\mu$ and $\Delta\sigma_\tau$

We now turn to the estimation of $\Delta\mu$ and $\Delta\sigma_\tau$. Using the full notation, we have that:

$$\Delta\mu = E\log(1+\tau(\boldsymbol{z_i};\boldsymbol{\Theta_1},w_1,Y_1)) - E\log(1+\tau(\boldsymbol{z_i};\boldsymbol{\Theta_0},w_0,Y_0))$$
$$\Delta\sigma_\tau = \mathrm{Var}\log(1+\tau(\boldsymbol{z_i};\boldsymbol{\Theta_1},w_1,Y_1)) - \mathrm{Var}\log(1+\tau(\boldsymbol{z_i};\boldsymbol{\Theta_0},w_0,Y_0))$$

where $\boldsymbol{z_i}$ is the entire past history of productivity shocks of firm i.

For now, we assume that we can observe the distribution of log wedges in the data (we return to measurement of wedges with CES technology below). Even then, the problem is that we do not directly observe these moments in the data. Assuming the experiment is small, the moments we observe are:

$$
\begin{aligned}
\widehat{\Delta\mu} &= E\log(1+\tau(\boldsymbol{z_i};\boldsymbol{\Theta_1},w_0,Y_0)) - E\log(1+\tau_i(\boldsymbol{z_i};\boldsymbol{\Theta_0},w_0,Y_0)) \\
\widehat{\Delta\sigma_\tau} &= \operatorname{Var}\log(1+\tau_i(\boldsymbol{z_i};\boldsymbol{\Theta_1},w_0,Y_0)) - \operatorname{Var}\log(1+\tau_i(\boldsymbol{z_i};\boldsymbol{\Theta_0},w_0,Y_0))
\end{aligned}
$$

or put differently: the data only show us the effect of the treatment on moments in *partial equilibrium*. To deal with this problem, we need to put structure on how wedges are affected by macro variables.

First, we show that these moments do *not* depend on aggregate demand Y (like in the Cobb-Douglas case):

Lemma A.2. *If the production function is homogeneous of degree 1, then the distribution of wedges does not depend on output Y.*

$$\tau(\boldsymbol{z_i};\boldsymbol{\Theta},w_0,Y) = \tau(\boldsymbol{z_i};\boldsymbol{\Theta},w_0,Y_0)\text{ , for all } Y$$

Proof. This property uses the degree one homogeneity of the production function. First, show that EBITDA can be rescaled by total output Y:

$$
\begin{aligned}
\pi(z,k;w,Y) &= \max_l \left(Y^{1-\theta}e^\theta z\,(F(k,l))^\theta - wl\right) \\
&= Y\max_l \left(e^\theta z\left(F\left(\frac{k}{Y},\frac{l}{Y}\right)\right)^\theta - w\frac{l}{Y}\right) \\
&= Y\max_{l'} \left(e^\theta z\left(F\left(\frac{k}{Y},l'\right)\right)^\theta - wl'\right) \\
&= Y\pi\left(z,\frac{k}{Y};w,Y_0\right)
\end{aligned}
$$

The rest of the proof then follows the logic of the proof of proposition 2. □

But the distribution of wedges and productivity may depend on the aggregate wage w. So we need to put structure on this dependence. The following Lemma does this:

Lemma A.3. *At the second order approximation, the following equations must hold:*

$$
\begin{aligned}
\Delta\mu &= \widehat{\Delta\mu} + \beta\Delta\log\frac{w_1}{w_0} + \gamma\Delta\left(\log\frac{w_1}{w_0}\right)^2 \\
\Delta\sigma^2 &= \widehat{\Delta\sigma^2} \\
\Delta\sigma_{z\tau} &= \widehat{\Delta\sigma_{z\tau}}
\end{aligned}
$$

β and γ can be obtained through regressing the wedges of treated firms *on exogeneous sources of variation of log wages.*

The two second order moments do not depend on the level of wages in the new aggregate counterfactual economy.

The coefficients β and γ can be estimated by regressing the log wedge of treated firms on the log local wage. It is important that the variation in log wages that is exploited in this regression is exogenous to the reform studied as for example in Oberfield and Raval (2014). For instance, imagine a cross-section of cities contains treated firms in all cities. We regress the log wedge of treated firms on city-level wages, *instrumented* by for instance shocks on local labor market supply, or shocks to local labor market demand. A valid instrument here would need to be unrelated to local intensity of the treatment that we study. For instance, if we study the effect of taxes, we can use deindustrialization Bartik shocks à la Oberfield and Raval (2014). The identifying assumption would be that local tax policy is independent of deindustrialization.

Proof. To prove the above result, we first need to log-linearize wedges with respect to w and z_i up to the second order and write:

$$\log(1+\tau(z_i,\Theta,w)) = \log(1+\tau(z_i,\Theta,w_0)) + \beta(z_i,\Theta,w_0)\log\left(\frac{w}{w_0}\right) + \frac{1}{2}\gamma(z_i,\Theta,w_0)\left(\log\left(\frac{w}{w_0}\right)\right)^2 + o(2)$$

The coefficients β and γ depend on the model generating the frictions which is generically described by Θ. We describe below how to estimate them from the data.

We then Taylor-expand both coefficients β and γ with respect to z_i around 0 (remember that $\mathbb{E}z_i = 0$). First order is enough:

$$\beta(z_i,\Theta,w_0) = \beta(0,\Theta,w_0) + \beta_1(0,\Theta,w_0)z_i + o(1)$$
$$\gamma(z_i,\Theta,w_0) = \gamma(0,\Theta,w_0) + \gamma_1(0,\Theta,w_0)z_i + o(1)$$

which leads to the following expression for wedges:

$$\log(1+\tau(z_i,\Theta,w)) = \log(1+\tau(z_i,\Theta,w_0)) + \beta(0,\Theta,w_0)\log\left(\frac{w}{w_0}\right) + \beta_1(0,\Theta,w_0)z_i\log\left(\frac{w}{w_0}\right) + \frac{1}{2}\gamma(0,\Theta,w_0)\left(\log\left(\frac{w}{w_0}\right)\right)^2 + o(2)$$

So we are now ready to compute the mean and variances of the wedge distribution. Let us start with the mean:

$$
\begin{aligned}
\mu(\Theta, w) &= \mathbb{E}\log(1+\tau(z_i,\Theta,w)) \\
&= \mu(\Theta, w_0) + \underbrace{\beta(0,\Theta,w_0)}_{\equiv\beta(\Theta,w_0))}\log\left(\frac{w}{w_0}\right) + \frac{1}{2}\underbrace{\gamma(0,\Theta,w_0)}_{\equiv\gamma(\Theta,w_0))}\left(\log\left(\frac{w}{w_0}\right)\right)^2
\end{aligned}
$$

which shows that the mean wedge is a function of $\log w$ and its square.

For the variance, we further need to expand:

$$
\log(1+\tau(z_i,\Theta,w_0) = \log(1+\tau(0,\Theta,w_0)) + T_1(0,\Theta,w_0)z_i + o(1)
$$

so that the variance writes:

$$
\begin{aligned}
\sigma_\tau^2(\Theta, w) &= \mathbf{var}\left(\log(1+\tau(z_i,\Theta,w))\right) \\
&= \sigma_\tau^2(\Theta, w_0) \\
&\quad + 2\underbrace{\mathbf{cov}\left(T_1(0,\Theta,w_0)z_i, \beta_1(0,\Theta,w_0)z_i\log\left(\frac{w}{w_0}\right)\right)}_{T_1(0,\Theta,w_0)\beta_1(0,\Theta,w_0)\sigma_z^2\log\left(\frac{w}{w_0}\right)=o(2)} \\
&\quad + \underbrace{\mathbf{var}\left(\beta_1(0,\Theta,w_0)z_i\log\left(\frac{w}{w_0}\right)\right)}_{=o(2)} \\
&= \sigma_\tau^2(\Theta, w_0) + o(2)
\end{aligned}
$$

which ensures that the variance of wedges does not depend on $\log w$ in our second order approximation.

Near identical algebra shows that $\sigma_{z\tau}(\Theta, w)$ is also independent from $\log w$ in our second order approximation.

We now compute the empirical moments as a function of observed sufficient statistics on wedges. Let us start again with the mean:

$$
\begin{aligned}
\Delta\mu &= \mu(\Theta_1, w_1) - \mu(\Theta_0, w_0) \\
&= (\mu(\Theta_1, w_1) - \mu(\Theta_1, w_0)) + \underbrace{(\mu(\Theta_1, w_0) - \mu(\Theta_0, w_0))}_{\widehat{\Delta\mu}} \\
&= \underbrace{\beta(\Theta_1, w_0))}_{\equiv\beta}\log\left(\frac{w_1}{w_0}\right) + \frac{1}{2}\underbrace{\gamma(\Theta_1, w_0))}_{\equiv\gamma}\left(\log\left(\frac{w_1}{w_0}\right)\right)^2 + \widehat{\Delta\mu}
\end{aligned}
$$

The coefficients B and C can be estimated by regressing the log wedge of treated firms on the log local wage. It is important that the variation in log wages that is exploited in this regression is exogenous to the reform studied as for example in Oberfield and Raval (2014). A cross-section of cities contains treated firms in all cities. We regress the log wedge of treated firms on local wages, *instrumented* by for instance shocks on local labor market supply, or shocks to local labor market demand that is unrelated to local intensity of the treatment that we study. For instance, if we study the effect of taxes, we can use deindustrialization Bartik shocks à la Oberfield and Raval (2014).

Assuming we know β and γ, we use the above formula for $\Delta\mu$ jointly with the previously derived formula on $\log w$:

$$\log\left(\frac{w}{w_0}\right) = \alpha_1\Delta\mu + \alpha_2(\Delta\mu)^2 + \underbrace{\alpha_3\Delta\sigma_\tau^2 + \alpha_4\Delta\sigma_{\tau,z}}_{X}$$

where the terms in X are not affected by the level of wages as we have just shown. Then:

$$\Delta\mu = \widehat{\Delta\mu} + \beta(\alpha_1\Delta\mu + \alpha_2(\Delta\mu)^2 + X) + \gamma(\alpha_1)^2(\Delta\mu)^2$$

Rearranging:

$$(\Delta\mu)^2\left(\beta\alpha_2 + \gamma(\alpha_1)^2\right) + \Delta\mu\left(B\alpha_1 - 1\right) + \beta X + \widehat{\Delta\mu} = 0$$

Now we let $\Delta\mu = \kappa_1\widehat{\Delta\mu} + \kappa_2\left(\widehat{\Delta\mu}\right)^2 + \kappa_3 X$ in the equation above. We obtain, by identification:

$$\begin{aligned}
\kappa_1 &= \frac{1}{1-\beta\alpha_1} = \frac{a}{a+b\beta}\\
\kappa_2 &= \frac{\beta\alpha_2 + \gamma(\alpha_1)^2}{(1-\beta\alpha_1)^3} = \frac{a\beta b}{(a+\beta b)^3}\left(\frac{b\gamma}{\beta} + \frac{1}{2}\frac{\rho}{1-\rho}\right)\\
\kappa_3 &= \frac{\beta}{1-\beta\alpha_1} = \frac{a\beta}{a+b\beta}
\end{aligned}$$

so that:

$$\Delta\mu = \frac{a}{a+b\beta}\widehat{\Delta\mu} + \frac{a\beta b}{(a+\beta b)^3}\left(\frac{b\gamma}{\beta} + \frac{1}{2}\frac{\rho}{1-\rho}\right)\left(\widehat{\Delta\mu}\right)^2 + \frac{a\beta}{a+b\beta}(\alpha_3\Delta\sigma_\tau^2 + \alpha_4\Delta\sigma_{\tau,z})$$

We are nearly done. Now, note that observed changed in variance is equal to the theoretical one given our second order aproximation:

$$\begin{aligned}
\Delta\sigma_\tau^2 &= \sigma_\tau^2(\Theta_1, w_1) - \sigma_\tau^2(\Theta_0, w_0)\\
&= \underbrace{\left(\sigma_\tau^2(\Theta_1, w_0) - \sigma_\tau^2(\Theta_0, w_0)\right)}_{\widehat{\Delta\sigma_\tau^2}}
\end{aligned}$$

Obviously, the same algebra works for $\Delta\sigma_{z\tau}$. $\square$

Given Lemma A.2 and A.3, we know how to compute $\Delta\mu, \Delta\sigma_\tau^2$, and $\Delta\sigma_{z\tau}$ as a function of the observed sufficient statistics $\widehat{\Delta\mu}$, $\widehat{\Delta\sigma_\tau^2}$, and $\widehat{\Delta\sigma_{z\tau}}$. We plug these into the formula of $\Delta\log Y$ and obtain:

$$\Delta \log Y = A\widehat{\Delta\mu} + B\widehat{\Delta\sigma_\tau^2} + C\widehat{\Delta\sigma_{\tau z}} + D(\widehat{\Delta\mu})^2$$

where:

$$\begin{aligned}
A &= -\frac{b}{a+\beta b}\left(\epsilon + \frac{1}{1-\rho}\right) \\
B &= \frac{b^2}{2}\frac{\rho(\theta-\rho)}{(1-\theta)(1-\rho)^2} + \frac{1}{2}\left(\epsilon + \frac{1}{1-\rho}\right)\frac{b}{a+b\beta}\left(\frac{\rho}{1-\rho}a + \frac{\theta}{1-\theta}b\right) \\
C &= -\frac{b}{a+b\beta}\left(\epsilon + \frac{1}{1-\rho}\right)\frac{\theta}{1-\theta} \\
D &= \frac{1}{2}\frac{b}{(a+b\beta)^3}\left(\epsilon + \frac{1}{1-\rho}\right)\left(a\frac{\rho}{1-\rho} - 2b^2\gamma\right)
\end{aligned}$$

A.8.4 Measuring wedges at the firm level

The last step consists of measuring wedges at the firm level in order to recover the three moments $\widehat{\Delta\mu}$, $\widehat{\Delta\sigma_{\tau z}}$ and $\widehat{\Delta\sigma_\tau^2}$. We start from the fact that firm-level labor and capital shares are given by:

$$\begin{aligned}
s_i^L &= \theta\frac{B}{A_i + B} \\
s_i^K &= \theta(1+\tau_i)\frac{A_i}{A_i + B}
\end{aligned}$$

where $B = (1-\alpha)\left(\frac{(1-\alpha)\theta}{w}\right)^{\frac{\rho}{1-\rho}}$ and $A_i = \alpha\left(\frac{\alpha\theta}{R(1+\tau_i)}\right)^{\frac{\rho}{1-\rho}}$.

Combining these two equations, we obtain that:

$$\log(1+\tau_i) = \log\left(\frac{s_i^K}{\theta - s_i^L}\right)$$

which gives a simple way to compute the distiortion at the firm level in the CES case. It converges to the formula used in the Cobb-Douglas case ($\frac{py}{k}$) wen ρ goes to zero. The downside of this formula is however that it requires to know the cost of capital R in order to compute the capital share, as well as the mark-up (θ).

A.8.5 Pulling it All Together

We combine all the above insights in the following summary:

1. We compute the following three sufficient statistics from the small-scale experiment:

$$\widehat{\Delta\mu} = E\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 1\right) - E\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 0\right)$$
$$\widehat{\Delta\sigma_\tau^2} = \text{Var}\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 1\right) - E\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 0\right)$$
$$\widehat{\Delta\sigma_{\tau z}} = \text{Cov}\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 1\right) - E\left(\log\left(\frac{s_i^K}{\theta - s_i^L}\right)|T_i = 0\right)$$

where s_i^L is the labor share at the firm level and s_i^K is the capital share.

2. Compute the following two aggregation sufficient statistics β and γ (not necessarily using the experiment) by running the regression, for firm i in economy c:

$$\log\left(\frac{s_i^K}{\theta - s_i^L}\right) = \textbf{cst} + \beta \log w_c + \gamma(\log w_c)^2 + u_i c$$

running this regression requires firm level data on labor and capital shares in a cross-section of economies (sectors, cities) and an exogenous source of variation for w.

3. Compute the additional parameter $a = \frac{s^L}{\theta}$ where s^L is the average labor share in the cross-section. Note $b = 1 - a$.

4. Then, the aggregate effect of the experiment would be given by:

$$\Delta \log Y = A\widehat{\Delta\mu} + B\widehat{\Delta\sigma_\tau^2} + C\widehat{\Delta\sigma_{\tau z}} + D(\widehat{\Delta\mu})^2$$

where A, B, C and D are known functions of a, β, γ, as well as θ, ϵ and ρ.

The formulas for A, B, C, D are given by:

$$A = -\frac{b}{a + \beta b}\left(\epsilon + \frac{1}{1-\rho}\right)$$
$$B = \frac{b^2}{2}\frac{\rho(\theta - \rho)}{(1-\theta)(1-\rho)^2} + \frac{1}{2}\left(\epsilon + \frac{1}{1-\rho}\right)\frac{b}{a + b\beta}\left(\frac{\rho}{1-\rho}a + \frac{\theta}{1-\theta}b\right)$$
$$C = -\frac{b}{a + b\beta}\left(\epsilon + \frac{1}{1-\rho}\right)\frac{\theta}{1-\theta}$$
$$D = \frac{1}{2}\frac{b}{(a + b\beta)^3}\left(\epsilon + \frac{1}{1-\rho}\right)\left(a\frac{\rho}{1-\rho} - 2b^2\gamma\right)$$

The first order version of the overall formula is much simpler, with:

$$\Delta \log Y = -\frac{\theta - s^L}{\beta\theta + s^L(1-\beta)}\left(\epsilon + \frac{1}{1-\rho}\right)\widehat{\Delta E \log\left(\frac{s_i^K}{\theta - s_i^L}\right)}$$

which shows quite clearly that the CES case requires three types of adjustment. First, under CES, distortions are affected by the GE of the economy: This is β. Second, under CES GE, effects themselves are more dampening than under Cobb-Douglas if $\rho < 0$, because labor cannot be substituted with capital as easily: This is $\frac{1}{1-\rho}$ instead of 1. Third, wedges cannot be computed using the sales to capital ratio as in the Cobb-Douglas case: This is the last term, which uses input shares to measure distortions.

Further, note that the GE effect is negligible iff:

$$\beta \ll \frac{s_L}{\theta - s_L} \approx \frac{.7}{.8 - .7} = 7$$

. This for instance the case in the simulation exercise below.

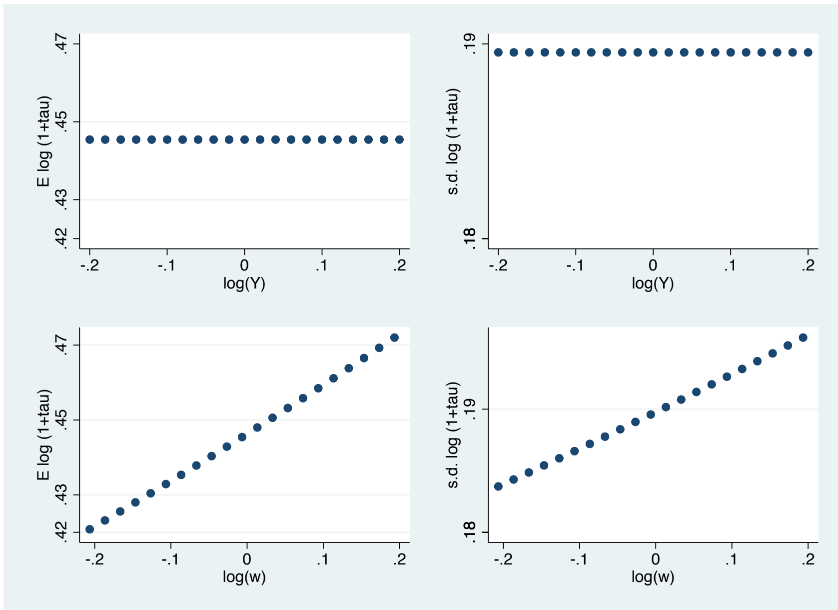
A.8.6 Simulation

In this Section, we simulate a simple version of our model and check that our CES formula is correct. We start from a version of our baseline model with no tax. Importantly, we allow firms to have a Cobb-Douglas production technology $y = e^z \left(\alpha k^\rho + (1-\alpha) l^\rho\right)^{1/\rho}$. There are quadratic adjustment costs $c\frac{I^2}{K}$. Firms cannot issue any equity, and can only borrow up to ζ times their expected capital stock k'. The rest follows exactly the baseline model described in Section 2. We simulate the model in equilibrium via iteration: We take (Y, w) as given, solve the Bellman problem, simulate a given number of firms in steady state, and then compute aggregate output Y^s and labor demand L^d. As long $Y \neq Y^s$ or $L^d \neq L^s(w)$, we change Y and w in order to reach equilibrium.

The parameters we pick are standard: capital share is $\alpha = .3$, CES aggregator substitutbility is $\theta = .85$, labor supply elasticity is $\epsilon = .5$. Rate of physical obsolescence is $\delta = .06$ and the safe rate of return is $r = .03$. Since we want to explore the effect of imperfect capital-labor substitution we vary ρ from -1.55 to -.05 (near Cobb-Douglas case). Elasticity of substitution of -.5 corresponds to $\rho = -1$.

We then consider the following hypothetical small scale experiment: We have data for firms whose collateral coefficient goes from $.2$ to $.25$. We simulate an economy in equilibrium with $\zeta = .2$ and obtain aggregate output Y_0. Given (w_0, Y_0), we then simulate the data on treated firms, for which $\zeta = .25$. We use this second sample to measure the effect of the treatment on various sufficient statistics in the context of a small scale experiment (sufficiently small so that (w, Y)) do not vary. We retrieve the sufficient statistics $\widehat{\Delta\mu}, \widehat{\Delta\sigma^2_\tau}, \widehat{\Delta\sigma_{\tau z}}$.

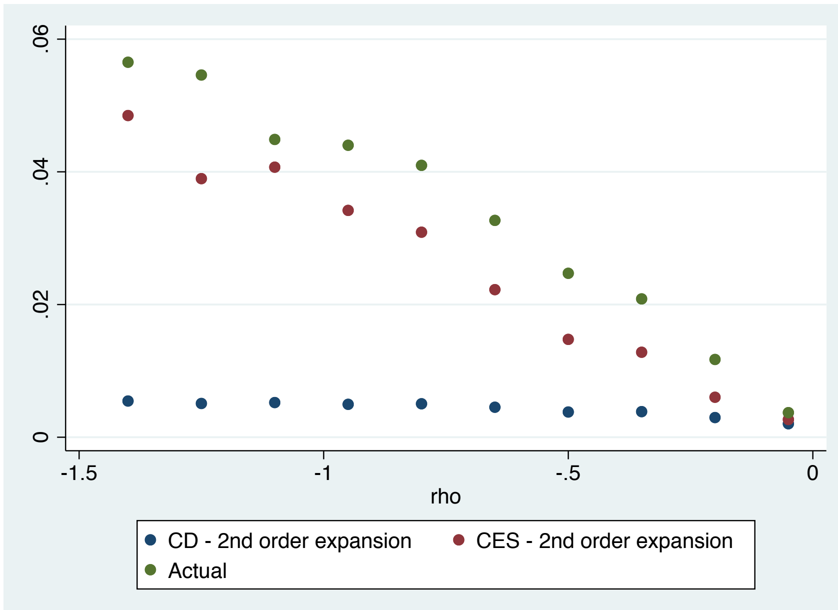
Figure A.1: Effect of Changing w and Y on $E\log(1+\tau_i)$



Source: Authors' simulations. In this simulation, we compute the mean log wedge in partial equilibrium. We normalize the exercise so that $Y = w = 1$ corresponds to the general equilibrium of this economy. Then, we vary Y or w around the equibrium, and show how the expected wedge responds.

We then apply our aggregation formula with these statistics. In a first pass, we assume $\beta = \gamma = 0$. This assumption is vindicated by our finding that, in this simulation at least, expected log wedge μ does not vary too much with $\log w$. We show this in Figure A.1: The mean log wedge does not vary with Y, as we showed previously. This is an analytical result. The mean log wedge does, however, vary with w. The elasticity implied by this graph is fairly constant (the log-log line is straight) and quite low (in terms of our notations, $\beta = .1$ and $\gamma = .025$). These elasticities are small here and we therefore neglect them. For instance iven the formula for B, the effect of β is negligible if $\beta \ll \frac{s_L}{\theta - s_L} \approx 7$. Hence, assuming $\beta \approx 0$ in the formula B is a good approximation.

Figure A.2: Effect on Output of Changing ζ from .2 to .25



Source: Authors' simulations.

Finally, to compute the terms A, B, C, D, we need to compute the parameter a which we approximate –as suggested by our analysis– by s^L/θ where s^L is the mean labor share.

We then apply our aggregation formula to the sufficient statstics computed above, and obtain aggregate output. We compare this number to the effective equilibrium output obtained in an economy simulated with $\zeta = .25$ in equilibrium. We report the results in Figure A.2. Our second order formula underestimates the macro effect by less than 1 percentage point, while the Cobb-Douglas formulas (the formulas in the main text) estimate almost no macro effect at all. The error implied by the Cobb-Douglas formula is bigger for lower values of ρ. For $\rho = -1$ – a conventional value of substitution elasticity – the Cobb Douglas formulas underestimate the effect of the reform on GDP by about 5 percentage points, while the CES formula is close to the true value. Comfortingly, all three values are very close to one another in the near Cobb-Douglas case of $\rho = -0.05$.

B Additional Results

B.1 Effect of Treatment on Productivity

Though we focus on treatment that purely affect wedges and not productivity, our framework can easily be extended to accomodate such settings. In this extension, we start from the baseline model but allow the treatment to affect the distribution of firm log productivities z. Like in the rest of the paper, we assume that z can be measured by the econometrician (TFP measurement is beyond the scope of the present paper). Such an extension could for instance apply to firm-level interventions designed to increase productivity via improved management practices Bloom et al. (2013a) or R&D subsidies (Bloom et al., 2013b). To clarify exposition, the present framework does not incorporate externalities but could easily be extended to do so – we leave such an important extension for future research.

The following proposition describes the aggregation procedure of an empirical micro treatment affecting both distortions and productivities:

Proposition B.1. *Assume the economy is described by the model of Section 2. Furthermore, assume the empirical treatment affects the joint distribution of distortions τ_i and productivities z_i. Then, the results of proposition 2 apply and the procedure to compute the aggregate counterfactual is as follows:*

1. *Estimate the following sufficient statistics resulting from the treatment:*

$$
\begin{aligned}
\widehat{\Delta\mu_\tau} &= \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=1\right) - \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=0\right)\\
\widehat{\Delta\sigma^2_\tau} &= \mathrm{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=1\right) - \mathrm{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=0\right)\\
\widehat{\Delta\sigma_{z\tau}} &= \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i=1\right) - \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i=0\right)\\
\widehat{\Delta\mu_z} &= \mathbb{E}\left(z_{it}|T_i=1\right) - \mathbb{E}\left(z_{it}|T_i=0\right)\\
\widehat{\Delta\sigma^2_z} &= \mathrm{Var}\left(z_{it}|T_i=1\right) - \mathrm{Var}\left(z_{it}|T_i=0\right)
\end{aligned}
$$

2. *Inject these sufficient statistics into the following aggregation formulas:*

$$
\begin{aligned}
\Delta\log Y &= \underbrace{\frac{1+\epsilon}{1-\alpha}\left(\widehat{\Delta\mu_z} + \frac{1}{2}\frac{\theta}{1-\theta}\widehat{\Delta\sigma^2_z}\right)}_{\textit{productivity effect}} + \underbrace{\frac{\alpha(1+\epsilon)}{1-\alpha}\left(-\widehat{\Delta\mu_\tau} + \frac{1}{2}\frac{\theta}{1-\theta}\left(\alpha\widehat{\Delta\sigma^2_\tau} - 2\widehat{\Delta\sigma_{\tau z}}\right)\right)}_{\textit{equation (8)}}\\
\Delta\log(TFP) &= \underbrace{\widehat{\Delta\mu_z} + \frac{1}{2}\frac{\theta}{1-\theta}\widehat{\Delta\sigma^2_z}}_{\textit{productivity effect}} - \underbrace{\frac{\alpha}{2}\left(1+\frac{\alpha\theta}{1-\theta}\right)\widehat{\Delta\sigma^2_\tau}}_{\textit{equation (7)}}
\end{aligned}
$$

The intuition of these extended formulas is straighforward. A new "productivity" term $\widehat{\Delta\mu_z} + \frac{1}{2}\frac{\theta}{1-\theta}\widehat{\Delta\sigma^2_z}$ is added to capture the direct effect of the treatment on firm productivity. This new term has two components: The effect of the treatment on mean log productivity plus the standard variance correction accounting for the concavity of the log function. This variance correction is necessary here because our sufficient statistics concern the distribution of log productivities, not the distribution of productivities. Our formulas could be written in terms of productivity levels, but in practice estimating logs yields more robust estimates. The productivity term enters the TFP aggregate directly, without GE effects since effects on productivity are not crowded out. It also enters the output formula, multiplied by a coefficient $\frac{1+\epsilon}{1-\alpha}$ designed to account for the fact that the quantities of labor and capital respond positively to productivity increases.

The above proposition highlights the importance of accounting for distortions even if the treatment is only supposed to affect productivity. Even if the treatment is not supposed to directly affect distorions (think for instance of an improvement in management practices or process innovation), it may *interact* with existing distortions (real or financial) and affect the distributions of τ_i's among treated firms. The above formulas make clear that the effect of the treatment on log TFP is not the only sufficient statistic to look at.

B.2 Decreasing returns to scale

In this Appendix, we extend our baseline model to allow for decreasing technological returns to scale (span of control): $y_{it} = e^{z_{it}}(k_{it}^{\alpha} l_{it}^{1-\alpha})^{\nu}$ where $\nu < 1$. The aggregation formulas are then similar to those in our baseline case, with minor modifications:

Proposition B.2 (Decreasing Returns to Scale)**.** *With decreasing technological returns to scale ν and under the homogeneity assumptions of Proposition 2, the joint-distribution of z and τ does not depend on (w, Y).*

With decreasing returns to scale, the aggregation formulas become:

$$\Delta \log(Y) = \frac{\alpha\nu\,(1+\epsilon)}{(1-\alpha)\nu + (1+\epsilon)(1-\nu)} \left(-\widehat{\Delta\mu_\tau} + \frac{1}{2}\frac{\nu\theta}{1-\nu\theta}\left(\alpha\widehat{\Delta\sigma_\tau^2} - 2\widehat{\Delta\sigma_{z\tau}}\right)\right)$$
$$\Delta \log(TFP) = -\frac{\alpha\nu}{2}\left(1 + \frac{\alpha\nu\theta}{1-\nu\theta}\right)\widehat{\Delta\sigma_\tau^2},$$

where $\widehat{\Delta\mu_\tau}$, $\widehat{\Delta\sigma_\tau^2}$, $\widehat{\Delta\sigma_{z\tau}}$ are the same treatment effects defined in Section 3.4.

Proof. See Appendix C.1. □

The modifications introduced by decreasing returns to scale are marginal. Proposition B.2 makes clear that our approach also applies to models of perfect competition ($\theta = 1$) and decreasing returns to scale such as Hopenhayn (2014) or Midrigan and Xu (2014). It also makes clear that the modifications induced by decreasing returns to scale $\nu < 1$ will quantitatively be small, since ν is typically close to 1. For instance, assuming $\alpha = .3$, $\epsilon = 1.5$ and $\nu = .95$, we find a pre-multiplying factor on output of .57 insead of .64.

B.3 Non-parametric Formulas

Here, we explore here the effect of relaxing the assumption of small variations in distortions and productivity. It turns out that simple formulas, similar to (7-8) can be developed. These formulas rely more heavily on the Cobb-Douglas nature of production, but present the advantage that they do not require the estimation of firm-level TFP shocks z. They rely on slightly different sufficient statistics than the sales to capital ratio.

Proposition B.3.
Consider the baseline framework of Section 2. Assume that the assumptions in Proposition 2 hold.

Define the following treatment effects for labor and capital:

$$\begin{cases} \widehat{\Delta l} = \ln\left(\mathbb{E}[l_{it}|T_i = 1\right) - \ln\left(\mathbb{E}[l_{it}|T_i = 0]\right) \\ \widehat{\Delta k} = \ln\left(\mathbb{E}[k_{it}|T_i = 1]\right) - \ln\left(\mathbb{E}[k_{it}|T_i = 0]\right) \end{cases}$$

Then, the effect of generalizing the treatment to all firms in the economy on aggregate output and TFP is given by the following formulas:

$$\begin{cases} \Delta \log Y = \dfrac{(1+\epsilon)(1-\theta)}{(1-\alpha)\theta} \times \widehat{\Delta l} \\ \Delta \log TFP = \left(\dfrac{1}{\theta} - (1-\alpha)\right) \times \widehat{\Delta l} - \alpha \times \widehat{\Delta k} \end{cases}$$

Proof. See Appendix C.2. □

The formulas in Proposition B.3 are intuitive and simply leverage the fact that reduction in distortions due to the treatment translates into changes in input use. The appeal of these formulas is that they do not require assumptions about the joint-distribution of productivity and wedges, or equivalently, assumption about the size of variations of productivity and wedges. However, these formulas may be unpractical from an empirical standpoint: with log-normally distributed labor and capital, estimating treatment effects in levels is likely to be inconsistent.

B.4 Exogenous Labor distortions

In this Appendix, we assume that the labor market also faces distortions. We further assume them to be exogeneous (like payroll taxes for example, which may vary across firms). Like for capital, we model them here as wedges between marginal productivities and the market wage w:

$$1 + \eta_{it} = (1-\alpha)\theta \frac{py_{it}}{wl_{it}}$$

This formulation leads to the following proposition:

Proposition B.4. *Assume that labor wedges are firm-specific but exogeneous. Furthermore, assume that their distribution is unaffected by the experiment – the experiment only affects capital distortions. To clarify exposition, we restrict ourselves here to treatment that do not affect labor wedges directly – our results are easy to extend to such a setting.*

Then, the results of proposition 2 apply and the procedure to compute the aggregate counterfactual is as follows:

1. *Estimate the following sufficient statistics resulting from the treatment:*

$$\widehat{\Delta\mu_\tau} = \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=1\right) - \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=0\right)$$
$$\widehat{\Delta\sigma^2_\tau} = \mathrm{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=1\right) - \mathrm{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=0\right)$$
$$\widehat{\Delta\sigma_{z\tau}} = \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i=1\right) - \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i=0\right)$$
$$\widehat{\Delta\sigma_{\eta\tau}} = \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), \log\left(\frac{p_{it}y_{it}}{l_{it}}\right)|T_i=1\right) - \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), \log\left(\frac{p_{it}y_{it}}{l_{it}}\right)|T_i=0\right)$$

2. *Inject these sufficient statistics into the following aggregation formulas:*

$$\Delta\log Y = \underbrace{-\frac{\theta}{1-\theta}\left(1+\alpha(1+\epsilon)\right)\widehat{\Delta\sigma_{\eta\tau}}}_{\textit{interaction effect}} + \underbrace{\frac{\alpha(1+\epsilon)}{1-\alpha}\left(-\widehat{\Delta\mu_\tau} + \frac{1}{2}\frac{\theta}{1-\theta}\left(\alpha\widehat{\Delta\sigma^2_\tau} - 2\widehat{\Delta\sigma_{\tau z}}\right)\right)}_{\textit{equation (8)}}$$
$$\Delta\log TFP = \underbrace{-\frac{\alpha(1-\alpha)\theta}{1-\theta}\widehat{\Delta\sigma_{\eta\tau}}}_{\textit{interaction effect}} + \underbrace{-\frac{\alpha}{2}\left(1+\frac{\alpha\theta}{1-\theta}\right)\widehat{\Delta\sigma^2_\tau}}_{\textit{equation (7)}}$$

Proof. See Appendix C.3. □

The above proposition simply states that the baseline formulas (7-8) need to be adjusted with a corrective term. This corrective term involves an additional statistic: The effect of the treatment on the covariance between labor and capital productivities. This is because of labor-capital complementarity: If the treatment increases capital distortions more for firms in which labor is heavily distorted, both output and productivity will be reduced more. This formula makes clear that, even if the treatment is not supposed to affect labor distortions directly, labor distortions, if they exist, affect the channel of transmission of the treatment since labor and capital are complement. Thus, the covariance between the effect on capital and labor wedges has to be included.

B.5 Labor distortions generated by binding minimum wages

In this Appendix, we focus on one particular type of distiortion: A binding minimum wage on low-skill labor. To investigate the effect of such distortions, we allow firms to hire both skilled and unsilled labor:

$$y_{it} = k_{it}^{\alpha} l_{s,it}^{\beta} l_{u,it}^{1-\alpha-\beta}$$

where l^s_{it} is the quantitiy of skilled labor, and l^u_{it} is the quantity of unskilled labor. Unskilled labor is subject to a binding minimum wage $\underline{w}$, but skilled labor is not, so that static labor optimization yields:

$$\beta\theta\frac{py_{it}}{l_{s,it}} = w$$
$$(1-\alpha-\beta)\theta\frac{py_{it}}{l_{u,it}} = \underline{w}_{it}w^{\zeta}$$

where w is the market clearing skilled wage. We allow the minimum wage to vary across firms (for instance due to collective agreements) and be indexed on the skilled wage with elasticity ζ.

In this economy, we can show that our baseline formulas are barely affected:

Proposition B.5. *Assume two types of workers: skilled and unskilled, which have a ES of one in production:*

$$y_{it} = k_{it}^{\alpha}(l_{s,it})^{\beta}(l_{u,it})^{1-\alpha-\beta}$$

Skilled labor market clears, but unskilled labor market faces a bindbing minimum wage. The minimum wage is indexed on skilled wage and can have an idiosyncratic component: $\underline{w}_{it}w^{\zeta}$.
Then, the results of proposition 2 apply and the procedure to compute the aggregate counterfactual is as follows:

1. *Estimate the following sufficient statistics resulting from the treatment:*

$$\widehat{\Delta\mu_{\tau}} = \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=1\right) - \mathbb{E}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=0\right)$$
$$\widehat{\Delta\sigma_{\tau}^2} = \mathrm{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=1\right) - \mathrm{Var}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right)|T_i=0\right)$$
$$\widehat{\Delta\sigma_{z\tau}} = \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i=1\right) - \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), z_{it}|T_i=0\right)$$
$$\widehat{\Delta\sigma_{\underline{w}\tau}} = \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), \underline{w}_{it}|T_i=1\right) - \mathrm{Cov}\left(\log\left(\frac{p_{it}y_{it}}{k_{it}}\right), \underline{w}_{it}|T_i=0\right)$$

2. *Inject these sufficient statistics into the following aggregation formulas:*

$$\Delta\log TFP = \underbrace{-\frac{\alpha}{2}\left(1+\frac{\alpha\theta}{1-\theta}\right)\widehat{\Delta\sigma_{\tau}^2}}_{\textit{equation (7)}}$$
$$+\underbrace{\frac{\theta\alpha(1-\alpha-\beta)}{1-\theta}\left[\widehat{\Delta\mu_{\tau}} - \frac{1}{2}\frac{\theta\alpha}{1-\theta}\widehat{\Delta\sigma_{\tau}^2} - \frac{\theta}{1-\theta}\widehat{\Delta\sigma_{z\tau}} - \left(1+\frac{\theta(1-\alpha-\beta)}{1-\theta}\right)\widehat{\Delta\sigma_{\underline{w}\tau}}\right]}_{\textit{correcting for minimum wage}}$$
$$\Delta\log Y = \frac{1-\alpha}{\beta+\zeta(1-\alpha-\beta)}\underbrace{\frac{\alpha(1+\epsilon)}{1-\alpha}\left(-\widehat{\Delta\mu_{\tau}} + \frac{1}{2}\frac{\theta}{1-\theta}\left(\alpha\widehat{\Delta\sigma_{\tau}^2} - 2\widehat{\Delta\sigma_{z\tau}}\right)\right)}_{\textit{equation (8)}}$$
$$-\underbrace{\frac{\alpha(1+\epsilon)}{\beta+\zeta(1-\alpha-\beta)}\frac{\theta(1-\alpha-\beta)}{1-\theta}\widehat{\Delta\sigma_{\underline{w}\tau}}}_{\textit{correcting for minimum wage}}$$

Proof. See Appendix C.4. □

The introduction of a minimum wage for unskilled workers adds a few corrective terms to our baseline. It disappears if there is no unskilled labor ($1 - \alpha - \beta = 0$). The two formulas require the calculation of an additional sufficient statistic: The covariance between log capital wedge and firm-level minimum wage. This statistic can also be measured indirectly as the covariance between capital and unskilled labor wedges. Output is reduced if the treatment increases capital distortions more in companies already facing a bigger minimum wage. The baseline effect of the treatment on output is also dampened by the indexation of the minimum wage on aggregate wages. The bigger indexation is (bigger ζ), the smaller the effect of the treatment on output will be.

C Proofs of Appendix Results

C.1 Proof of Proposition B.2

We first show that Proposition 2 still holds with decreasing returns to scale.

With monopolistic competition and decreasing returns to scale, for a firm i with a stock of capital k_{it}, operating profits after optimizing labor demand are given by:

$$\begin{aligned} p_i y_i - w l_i &= (1-(1-\alpha)\nu\theta)\left(\frac{(1-\alpha)\nu\theta}{w}\right)^{\frac{(1-\alpha)\nu\theta}{1-(1-\alpha)\nu\theta}} Y^{\frac{1-\theta}{1-(1-\alpha)\nu\theta}} e^{z_i \frac{\theta}{1-(1-\alpha)\nu\theta}} k_i^{\frac{\alpha\nu\theta}{1-(1-\alpha)\nu\theta}} \\ &= S \times (1-(1-\alpha)\nu\theta)\left((1-\alpha)\nu\theta\right)^{\frac{(1-\alpha)\nu\theta}{1-(1-\alpha)\nu\theta}} e^{z_i \frac{\theta}{1-(1-\alpha)\nu\theta}} \left(\frac{k_i}{S}\right)^{\frac{\alpha\nu\theta}{1-(1-\alpha)\nu\theta}} \end{aligned}$$

where $S = \frac{Y^{\frac{1-\theta(1-\alpha\nu)}{1-(1-\alpha)\nu\theta}}}{w^{\frac{\nu\theta}{1-(1-\alpha)\nu\theta}}}$.

It follows directly from the proof of Proposition 2 that in this economy, and under the assumptions of Proposition 2, the ergodic joint distribution of capital wedges and productivity is independent of (w, Y) and depend only on the parameters Θ. Let $F(z, \tau; \Theta)$ denote this distribution as before.

With decreasing returns to scale ν, profit maximization for firm i in industry s as a function of a capital wedge τ_{is} leads to:

$$\begin{cases} k_i \propto \left(\frac{1}{w}\right)^{\frac{\nu(1-\alpha)\theta}{1-\nu\theta}} Y^{\frac{1-\theta}{1-\nu\theta}} e^{z_i \frac{\theta}{1-\nu\theta}} \left(\frac{1}{(1+\tau_i)}\right)^{\frac{1-(1-\alpha)\nu\theta}{1-\nu\theta}} \\ l_i \propto \left(\frac{1}{w}\right)^{\frac{1-\alpha\nu\theta}{1-\theta}} Y^{\frac{1-\theta}{1-\nu\theta}} e^{z_i \frac{\theta}{1-\nu\theta}} \left(\frac{1}{(1+\tau_i)}\right)^{\frac{\alpha\nu\theta}{1-\nu\theta}} \end{cases}$$

Firm i output at the optimum is given by:

$$p_i y_i \propto Y^{\frac{1-\theta}{1-\nu\theta}} \left(\frac{1}{w}\right)^{\frac{(1-\alpha)\nu\theta}{1-\nu\theta}} \left(\frac{1}{1+\tau_i}\right)^{\frac{\alpha\nu\theta}{1-\nu\theta}} e^{z_i \frac{\theta}{1-\nu\theta}} \tag{17}$$

Omiting the i subscripts, equilibrium on the product market implies that:

$$w \propto Y^{-\frac{1-\nu}{(1-\alpha)\nu}} \left(\int_{z,\tau} \frac{e^{z\frac{\theta}{1-\nu\theta}}}{(1+\tau)^{\frac{\alpha\nu\theta}{1-\nu\theta}}} dF(z,\tau;\Theta)\right)^{\frac{1-\nu\theta}{(1-\alpha)\nu\theta}} \tag{18}$$

Equilibrium on the labor market implies that $Y \propto w^{1+\epsilon}$

Combining these two equations provides the following expression for aggregate output:

$$Y \quad \propto \quad \left(\int_{(z,\tau)} \frac{e^{z\frac{\theta}{1-\nu\theta}}}{(1+\tau)^{\frac{\alpha\nu\theta}{1-\nu\theta}}} dF(z,\tau;\Theta)\right)^{\frac{(1+\epsilon)(1-\nu\theta)}{\theta((1-\alpha)\nu+(1+\epsilon)(1-\nu))}}$$

which then leads to the expression in the proposition after Taylor expansion.

Finally, aggregate TFP admits a simple expression:

$$
\begin{aligned}
TFP &= \frac{Y}{K^{\alpha\nu}L^{(1-\alpha)\nu}} \\
&= \left(\int_{z,\tau} \frac{z^{\frac{\theta}{1-\nu\theta}}}{(1+\tau)^{\frac{\alpha\nu\theta}{1-\nu\theta}}} dF(z,\tau;\Theta)\right)^{\frac{1-(1-\alpha)\nu\theta}{\theta}} \left(\int_{z,\tau} \frac{z^{\frac{\theta}{1-\nu\theta}}}{(1+\tau)^{\frac{1-(1-\alpha)\nu\theta}{1-\nu\theta}}} dF(z,\tau;\Theta)\right)^{-\alpha\nu}
\end{aligned}
$$

which then leads to the formula in the proposition after straightforward Taylor expansion.

C.2 Proof of Proposition B.3

Optimal labor demand as a function of firm-level capital wedge is:

$$
l_i \propto \left(\frac{1}{w}\right)^{\frac{1-\alpha\theta}{1-\theta}} Y e^{\frac{\theta}{1-\theta}z_i} \left(\frac{1}{(1+\tau_i)}\right)^{\frac{\alpha\theta}{1-\theta}}
$$

Assume treated firms are a zero-measure set. Then, the following sufficient statistic can be computed for both the treatment and control groups:

$$
\mathbb{E}[l_i|T_i = T] \propto \int_{z,\tau} \frac{e^{\frac{\theta}{1-\theta}}}{(1+\tau)^{\frac{\alpha\theta}{1-\theta}}} dF(z,\tau,\Theta_T)
$$

where $T \in \{0,1\}$.

We now introduce the log difference in mean employment:

$$
\begin{aligned}
\widehat{\Delta l} &= \log\left(\mathbb{E}[l_i|T_i = 1]\right) - \log\left(\mathbb{E}[l_i|T_i = 0]\right) \\
&= \log \int_{z,\tau} \frac{e^{\frac{\theta}{1-\theta}}}{(1+\tau)^{\frac{\alpha\theta}{1-\theta}}} dF(z,\tau,\Theta_1) - \log \int_{z,\tau} \frac{e^{\frac{\theta}{1-\theta}}}{(1+\tau)^{\frac{\alpha\theta}{1-\theta}}} dF(z,\tau,\Theta_0)
\end{aligned}
$$

Given the output equation (3), it follows directly that

$$
\Delta \log Y = \frac{(1+\epsilon)(1-\theta)}{(1-\alpha)\theta} \widehat{\Delta_l}
$$

We now compute TFP, which requires calculating the capital stock. Similarly, optimal capital demand implies that:

$$
k_i \propto \left(\frac{1}{w}\right)^{\frac{(1-\alpha)\theta}{1-\theta}} Y e^{\frac{\theta}{1-\theta}z_i} \left(\frac{1}{1+\tau_i}\right)^{\frac{1-(1-\alpha)\theta}{1-\theta}}
$$

Like for employment, we use this to compute the new capital sufficient statistic:

$$
\begin{aligned}
\widehat{\Delta k} &= \log\left(\mathbb{E}[k_i|T_i = 1]\right) - \log\left(\mathbb{E}[k_i|T_i = 0]\right) \\
&= \log \int_{z,\tau} \frac{e^{\frac{\theta}{1-\theta}}}{(1+\tau)^{1+\frac{\alpha\theta}{1-\theta}}} dF(z,\tau,\Theta_1) - \log \int_{z,\tau} \frac{e^{\frac{\theta}{1-\theta}}}{(1+\tau)^{1+\frac{\alpha\theta}{1-\theta}}} dF(z,\tau,\Theta_0)
\end{aligned}
$$

Given the TFP formula (4), $\widehat{\Delta l}$ and $\widehat{\Delta k}$ can be straightforwardly combined into the formula given in the proposition.

C.3 Proof of Proposition B.4

Noting η_{it} the wedge between the marginal product of labor and the wage, and assuming that capital and labor wedges, as well as productivity, has small deviation around their means, we obtain that:

$$
\begin{aligned}
c^{\frac{\theta}{1-\theta}} &= I_Y \\
c^{\frac{\theta}{1-\theta}} wL &= (1-\alpha)\theta Y I_L \\
c^{\frac{\theta}{1-\theta}} RK &= \alpha\theta Y I_K
\end{aligned}
$$

where:

$$
\begin{aligned}
I_Y &= \int e^{\frac{\theta}{1-\theta} z_i} \left(\frac{1}{1+\tau_i}\right)^{\frac{\alpha\theta}{1-\theta}} \left(\frac{1}{1+\eta_i}\right)^{\frac{(1-\alpha)\theta}{1-\theta}} di \\
I_L &= \int e^{\frac{\theta}{1-\theta} z_i} \left(\frac{1}{1+\tau_i}\right)^{\frac{\alpha\theta}{1-\theta}} \left(\frac{1}{1+\eta_i}\right)^{\frac{(1-\alpha)\theta}{1-\theta}+1} di \\
I_K &= \int e^{\frac{\theta}{1-\theta} z_i} \left(\frac{1}{1+\tau_i}\right)^{\frac{\alpha\theta}{1-\theta}+1} \left(\frac{1}{1+\eta_i}\right)^{\frac{(1-\alpha)\theta}{1-\theta}} di
\end{aligned}
$$

Using a second order approximation or a log-normal assumption on z_i, τ_i and η_i, we obtain the following formulas for TFP and output:

$$
\begin{aligned}
\log TFP &= \left(\mu_z + \frac{1}{2}\frac{\theta}{1-\theta}\sigma_z^2\right) - \frac{\alpha}{2}\left(\sigma_\tau^2 + \sigma_\eta^2\right) - \frac{1}{2}\frac{\theta}{1-\theta}\text{var}(\Delta) \\
\log Y &= -\left(\mu_\eta + \frac{\sigma_\eta^2}{2}\right) + \frac{1+\epsilon}{1-\alpha}\left(-\mu_z + E\Delta + \frac{1}{2}\frac{\theta}{1-\theta}\text{var}(z-\Delta)\right) + \frac{\theta}{1-\theta}\text{cov}\left(z-\Delta, \eta\right)
\end{aligned}
$$

where we note $\mu_\eta = E\log(1+\eta)$, $\sigma_\eta^2 = \text{var}\log(1+\eta)$ and $\Delta = \alpha\log(1+\tau) + (1-\alpha)\log(1+\eta)$.

We then assume that the mean and variances of z and η are unaffected by the experiment, which gives the results in the proposition.

C.4 Proof of Proposition B.5

Scale invariance of capital wedges

Omitting the i, t subscripts, operating profits are given by:

$$\pi = \max_{l_s, l_u} \left(Y^{1-\theta} e^{\theta z} l_s^{\beta\theta} l_u^{(1-\alpha-\beta)\theta} - w l_s - \underline{w} w^{\zeta} \right)$$

which rewrites:

$$\pi = (1-\theta(1-\alpha)) Y^{\frac{1-\theta}{1-\theta(1-\alpha)}} e^{\frac{\theta z}{1-\theta(1-\alpha)}} \left(\left(\frac{\theta\beta}{w} \right)^{\beta} \left(\frac{\theta(1-\alpha-\beta)}{\underline{w} w^{\zeta}} \right)^{1-\alpha-\beta} \right)^{\frac{\theta}{1-\theta(1-\alpha)}} k^{\frac{\theta\alpha}{1-\theta(1-\alpha)}}$$

From the above formula, it appears easily that the firm problem given Y, w can be written as a firm problem given $1, 1$ scaled by:

$$S = \frac{Y}{w^{\frac{\theta(\beta+\zeta(1-\beta-\alpha))}{1-\theta}}}$$

so that the distribution of capital wedges is unaffected by the equilibrium. Note that the scaling parameter is the same as in proposition 2 when $\zeta = 0$.

Equilibrium formulas

We use the following notations:

$$\begin{aligned}
\chi &= \frac{\theta}{1-\theta} (\beta + \zeta(1-\alpha-\beta)) \\
u_{it} &= \frac{\theta}{1-\theta} (z_{it} - \alpha \log(1+\tau_{it}) - (1-\alpha-\beta) \log \underline{w}_{it}) \\
v_{it} &= u_{it} - \log(1+\tau_{it}) \\
\omega_{it} &= u_{it} - \log \underline{w}_{it}
\end{aligned}$$

Then, the three market clearing conditions write:

$$\begin{aligned}
w^{\chi} &\propto Eu \\
K &\propto \frac{1}{w^{\chi}} Ev \\
w L^s Y & \\
w^{\zeta} L^u &\propto Y E\omega
\end{aligned}$$

After some manipulation, we get the expression for TFP:

$$\Delta \log TFP = \underbrace{-\frac{\alpha}{2}\left(1 + \frac{\alpha\theta}{1-\theta}\right)\sigma_{\tau}^2}_{\text{equation (7)}} + \underbrace{\frac{\theta\alpha(1-\alpha-\beta)}{1-\theta}\left[\mu_{\tau} - \frac{1}{2}\frac{\theta\alpha}{1-\theta}\sigma_{\tau}^2 - \frac{\theta}{1-\theta}\sigma_{z\tau} - \left(1 + \frac{\theta(1-\alpha-\beta)}{1-\theta}\right)\sigma_{\underline{w}\tau}\right]}_{\text{correcting for minimum wage}}$$

omitting the terms in $\mu_{\underline{w}}$, $\sigma_{\underline{w}}^2$, μ_z and σ_z^2. These terms are assumed to be unaffected by the experiment.

The formula for log output is even simpler:

$$\log Y = \frac{\alpha(1+\epsilon)}{\beta + \zeta(1-\alpha-\beta)} \left(\underbrace{-\mu_\tau + \frac{1}{2}\frac{\theta}{1-\theta}\left(\alpha\sigma_\tau^2 - 2\sigma_{\tau z}\right)}_{\text{equation (8)}} - \underbrace{\frac{\theta(1-\alpha-\beta)}{1-\theta}\sigma_{\underline{w}\tau}}_{\text{correcting for minimum wage}} \right)$$