Problem 1

(a) The region where the joint density is \( c \) is a triangle of height 1 and base 2, hence

\[
\int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy = c \times \frac{1}{2} \times 1 \times 2 = 1,
\]

so that \( c = 1 \).

For \(-1 \leq x \leq 0\),

\[
f_X(x) = \int_0^{1+x} f_{X,Y}(x, y) \, dy = 1 + x.
\]

For \(0 \leq x \leq 1\),

\[
f_X(x) = \int_0^{1-x} f_{X,Y}(x, y) \, dy = 1 - x.
\]

We note that \( f_X(x) \) is zero outside \(-1 \leq x \leq 1\).

We find \( f_Y(y) \) as

\[
f_Y(y) = \int_{y-1}^{1-y} f_{X,Y}(x, y) \, dx = 2(1 - y), \quad 0 \leq y \leq 1,
\]

and 0 otherwise.

\( X \) and \( Y \) are not independent as \( f_{X,Y}(x, y) \) is not equal to \( f_X(x)f_Y(y) \).

(b) This probability is the integral of the joint density over a triangle with base (0,1) and a height of 1/3 at its peak where \( x = 2/3 \). \( (y = 1/3 \text{ is where the line } y = x/2 \text{ crosses the line } x + y = 1.)\)

This triangle has area

\[
\frac{1}{2} \times \frac{1}{3} \times 1 = \frac{1}{6}.
\]

(c) This is the area of the section of the triangle where the joint pdf is nonzero that lies between the lines \( x + y = 1/2 \) and \( x + y = 1 \). This is the area of the original triangle minus the area of the remaining triangle when the above region is removed, which is a triangle with base \((-1,1/2)\)
and height 3/4 at its peak at \( x = -1/4 \). In particular, \( x = -1/4 \) is the value of \( x \) where the line \( x + y = 1/2 \) intersects the line \( y - x = 1 \) and \( y = 3/4 \) at the intercept. So the desired area is

\[
1 - \frac{1}{2} \times \frac{3}{2} \times \frac{3}{4} = \frac{7}{16}.
\]

(d) The conditional pdfs can be evaluated as

\[
f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{2(1 - y)}, \quad |x| \leq 1 - y, \ y \in (0, 1),
\]
\[
f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{1 - |x|}, \quad 0 < y < 1 - |x|, |x| \leq 1.
\]

**Problem 2**

(a) To find the conditional pdf of \( P \), apply Bayes rule for mixed random variables.

\[
f_{P|X}(p|x) = \frac{P_{X|P}(x|p)}{\int_0^1 P_{X|P}(x|p)f_P(p)\,dp}f_P(p).
\]

Now it is given that \( X = 9 \). Therefore if \( 0 \leq p \leq 1 \) then

\[
f_{P|X}(p|9) = \frac{p^9(1 - p)}{\int_0^1 p^9(1 - p)\,dp} = \frac{p^9(1 - p)}{1/10 - 1/11} = \frac{p^9(1 - p)}{1/110} = 110p^9(1 - p).
\]

Given the information that 10 tosses resulted in 9 heads, the pdf is shifted towards the value 0.9.

(b) Let the result of the \( i \)-th toss be the random variable \( Y_i \), where \( Y_i = 1 \) corresponds to heads and \( Y_i = 0 \) corresponds to tails. The conditional pmf of \( Y_i \) given \( P = p \) is

\[
p_{Y_i|P}(y_i|p) = \begin{cases} p & \text{if } y_i = 1 \\ 1 - p & \text{if } y_i = 0. \end{cases}
\]

The tosses of the coin are independent given the bias, that is,

\[
p_{Y_1, \ldots, Y_{10}|P}(y_1, \ldots, y_{10}|p) = p_{Y_1|P}(y_1|p)p_{Y_2|P}(y_2|p) \cdots p_{Y_{10}|P}(y_{10}|p).
\]

If the order of the outcomes is nine heads followed by one tail, then

\[
p_{Y_1, \ldots, Y_{10}|P}(H, H, \ldots, H, T|P) = p^9(1 - p).
\]

So we use Bayes rule to find the conditional pdf of \( P \):

\[
f_{P|Y_1, \ldots, Y_{10}}(p|H, H, \ldots, H, T) = \frac{P_{Y_1,Y_2,\ldots,Y_{10}|P}(H, H, \ldots, H, T|P)}{P_{Y_1,Y_2,\ldots,Y_{10}}(H, H, \ldots, H, T)}f_P(p).
\]

This expression is zero when \( p < 0 \) or \( p > 1 \) since the a priori pdf \( f_P(p) \) is zero. For \( 0 \leq p \leq 1 \):

\[
f_{P|Y_1, \ldots, Y_{10}}(p|H, H, \ldots, H, T) = \frac{p^9(1 - p)}{\int_0^1 f_{Y_1,Y_2,\ldots,Y_{10}|P}(H, H, \ldots, H, T|p)f_P(p)\,dp} = \frac{p^9(1 - p)}{110p} = 110p^9(1 - p).
\]
Note that the result is independent of the order of heads and tails. This is due to the fact that the tosses are independent, and therefore the conditional pmf of the $Y_i$'s given $P$ is a function only of the number of heads.

**Problem 3**

(a) The region is two triangles with area $\frac{1}{2} \times 1 \times 2 = 1$. So $c = 1/2$

(b) No. Since

$$P(X \in [1/2, 1]) > 0, \quad P(Y \in (0, 1/3)) > 0$$

But $|X| \leq |Y|$, so

$$P(X \in [1/2, 1], Y \in (0, 1/3)) = 0$$

(c) Yes. Using the symmetry of the pdf, it is easy to verify that

$$\mathbb{E}X = 0, \quad \mathbb{E}Y = 0, \quad \mathbb{E}XY = 0$$

so $\mathbb{E}XY = \mathbb{E}XEY$

**Problem 4**

(a) The pdf of $Y$ is

$$f_Y(y) = \int_0^1 (x + y)dx = 1/2 + y$$

for $y \in [0, 1]$. So the conditional pdf of $X|Y$ is

$$f_{X|Y}(x|y) = \frac{x + y}{1/2 + y}$$

Then

$$\mathbb{E}(X|Y = y) = \int_0^1 \frac{x(x + y)}{1/2 + y}dx = \frac{2 + 3y}{3 + 6y}$$

i.e.

$$\mathbb{E}(X|Y) = \frac{2 + 3Y}{3 + 6Y}$$

(b) $Z = \mathbb{E}(X|Y) = \frac{2 + 3Y}{3 + 6Y}$, then

$$\left|\frac{dz}{dy}\right| = \frac{1}{3(1 + 2y)^2}, \quad y = \frac{3z - 2}{3 - 6z}$$

from $y \in [0, 1]$ we could derive that $z \in \left[\frac{5}{9}, \frac{2}{3}\right]$. Finally using $f_Z(z)\left|\frac{dz}{dy}\right| = f_Y(y)$ to get

$$f_Z(z) = \frac{3(1 + 2y)^2}{2} = \frac{3}{2} \left(\frac{1}{6z - 3}\right)^3, \quad z \in \left[\frac{5}{9}, \frac{2}{3}\right]$$