Solutions should be complete and concisely written. Please, mark clearly the beginning and end of each problem.

**You have 2 hours. Try to solve as many problems as you can during this time, but keep in mind that you can also get a good grade by solving a subset of problems.**

Points assigned to each problem are indicated in parenthesis. I recommend to look at all problems before starting.

For any clarification on the text, the TAs will be outside the room, and Andrea in Packard 272.

You can consult the Bertsekas and Tsitsiklis textbook, the reader and the lecture notes. You cannot consult other books, use computers, and in particular you cannot use the web.

Solutions should be written on the blue books. Please, write your name on each of the books.

**Problem 1** (10 points)

A box contains 6 balls. Two of them are red, two blue and two green. We extract the balls uniformly at random **without replacement**. This means that we extract the balls one the time, at random, without putting back in the box the balls extracted.

We let $X$ denote the extraction number at which draw for the first time a ball of a color that has been already extracted. For instance, if both the first and the second balls are red, then $X = 2$. If the first ball is red, but both the second and the third are blue, then $X = 3$.

(a) Define a sample space $\Omega$ and a probability law $P$ to model this experiment. Give a formal definition of the random variable $X(\omega)$. [Hint: A good option for the sample space has cardinality $|\Omega| = 6!$.]

(b) Compute the pmf of $X$, $p_X$. Plot a sketch of the function $p_X(k)$ versus $k$.

(c) Compute the mean of the pmf $p_X$.

(d) Let $A$ be the event that the first ball extracted is red, and $B$ the event that $X(\omega) = 3$. Are the events $A$ and $B$ independent? Justify your answer.

**Solution**

(a) Sample space: $\Omega = \{\omega = (\omega_1, ..., \omega_6) : \omega_i \in \{r_1, r_2, g_1, g_2, b_1, b_2\}, \omega_i \neq \omega_j \text{ for } i \neq j\}$

$$P(\{\omega\}) = 1/6! \forall \omega \in \Omega$$

$$X(\omega) = \min\{i : \text{ there exists } j \text{ s.t. } \omega_i \text{ and } \omega_j \text{ are of the same color}\}$$

Remark: It is important to include $\omega_i \neq \omega_j$ in the definition of $\Omega$, otherwise it is an extraction with replacement and $|\Omega| = 6$.

(b) It is easy to see $2 \leq X \leq 4$.

$$P(X = 2) = 3 \times 2 \times 4! / 6! = 1/5$$

$$P(X = 3) = 2 \times 3 \times 2 \times 4! / 6! = 2/5$$

$$P(X = 4) = 2/5$$

(c) $E[X] = 16/5$

(d) They are independent. $P(A) = 2/6$, $P(B) = P(X = 3) = 2/5$, and $P(A, B) = 2 \times 4! / 6!$ (the third is red) + $2 \times 2 \times 2 \times 3! / 6!$ (the third is not red) = 2/15. So $P(A, B) = P(A)P(B)$.
**Problem 2** (6 points)

A fair die is rolled repeatedly. Let $X$ be the random variable counting the number of rolls before the first six. For instance, if the first roll is a six, then $X = 0$. If the second roll is a six, then $X = 1$, and so on.

(a) Compute the pmf of $X$, i.e. give a formula for $p_X(k) \equiv \mathbb{P}(\{\omega : X(\omega) = k\})$ for each $k$.

(b) Let $Y$ be the random variable counting the number of rolls before the second six. For instance, if the outcomes are $3, 6, 5, 1, 6, \ldots$, then $Y = 4$. Compute the pmf of $Y$.

(c) Is $X$ independent of $Y$? Justify your answer.

**Solution**

(a) $p_X(k) = (5/6)^k \times 1/6$, $k \geq 0$.

(b) $P(Y = k) = k(5/6)^{k-1}(1/6)^2$.

(c) Not independent: $P(X = 3, Y = 3) = 0$ but $P(X = 3) \neq 0$, $P(Y = 3) \neq 0$.

**Problem 3** (15 points)

You have to hire a secretary and scheduled an large number of interviews with applicants (one interview per applicant). Call $n$ the number of applicants. However, you want to stop before interviewing everybody, as soon as you find a reasonable candidate.

You devise the following scheme. Candidates will be interviewed following to a uniformly random ordering. You rate each candidate with a real number between 0 and 1 (0 corresponding to a very bad candidate, and 1 to an outstanding one). We assume that the rating given to a candidate is a true reflection of his/her skills and that no two candidates have exactly the same skills (i.e. exactly the same rating). When you encounter the first candidate that is worse than the previous one, you stop and hire the best candidate interviewed so far (that is, the one before the last).

If the candidates are ordered so that each one is better than the previous one, then you hire the last.

We are interested in understanding how many interviews will be done. We let $X_n$ be the random variable giving the number of interviews done before stopping.

(a) Call $r_1 < r_2 < \cdots < r_n$ the ratings of the $n$ candidates where subscripts were chosen so that they are ordered in ascending order. Call $i \in \{1, \ldots, n\}$ the candidate with rating $r_i$. Define a sample space $\Omega$ and a probability law $\mathbb{P}$ that models the above setting.

(b) What is the probability that you will have to do $n$ interviews?

(c) For $k \in \{1, \ldots, n\}$ what is the probability that the first $k$ candidate are each better than the previous one?

(d) Let $A_k$ be the event introduced at the previous point. Show that $\mathbb{P}(\{\omega : X_n(\omega) > k\}) = \mathbb{P}(A_k)$. Compute the pmf of $X_n$, $p_{X_n}$.

(e) Assume that the number of candidates goes to infinity: $n \to \infty$. Compute the limit $p(k) = \lim_{n \to \infty} p_{X_n}(k)$ for each $k$. Is $p(\cdot)$ a pmf? Compute its mean.

**Solution**

(a) $\Omega = \{\tau : \tau$ is a permutation on $\{1,2,\ldots,n\}\}$, $P(\{\tau\}) = 1/n!$.

(b) The last one is arbitrary, the first $n - 1$ should be each better than previous. In particular, the order could be $r_2, r_3, \ldots, r_n, r_1$. Therefore $P(n$ interviews$) = n/n! = 1/(n - 1)!$. 


(c) Choose $k$ from $n$ to be the first $k$ candidates, and put them in ascending order, the remaining $n - k$ could be any arbitrary order. So:

$$P(A_k) = \binom{n}{k} (n-k)!/n! = 1/k!$$

(d) That the first $k$ candidates are each better than the previous means we should do more than $k$ interviews to hire a person. So:

$$P(A_k) = P(\{\omega: X_n(\omega) > k\})$$

Based on (c),

$$P(X_n = k) = P(X_n > k - 1) - P(X_n > k) = P(A_{k-1}) - P(A_k) = 1/(k-1)! - 1/k!$$

for $k < n$, from (b) we see

$$P(X_n = n) = 1/(n-1)!$$

(e) For $n > k$, $p_{X_n}(k) = 1/(k-1)! - 1/k!$, independent of $n$. Hence $p(k) = 1/(k-1)! - 1/k!$.

Note that $p(k) \geq 0$ and $\sum_{k=1}^{\infty} p(k) = 1$, so it is indeed a pmf. Its mean is

$$\sum_{k=1}^{\infty} kp(k) = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} = e$$