Problem 1
Which of the following set functions are probability measures? (You must justify your answers.)

1. \( \Omega = [0,1] \) with \( \mathbb{P} \) defined by
   \[
   \mathbb{P}(F) = \begin{cases} 
   \frac{1}{2} & \text{if } 0 \in F \text{ or } 1 \in F \text{ but not both} \\
   1 & \text{if } 0 \in F \text{ and } 1 \in F \\
   0 & \text{otherwise}
   \end{cases}
   \]

2. \( \Omega = \{1,2,3,4,5,6\} \)
   \[
   \mathbb{P}(F) = \frac{1}{C} \sum_{i \in F} i^2
   \]
   where \( C \) is a fixed constant which can be chosen.

3. \( \Omega = \text{real line} \)
   \[
   \mathbb{P}(F) = \int_F \sin x \, dx.
   \]

Problem 2
The probability that a man has a particular disease is 1/20. John is tested for the disease but the test is not totally accurate. The probability that a person with the disease tests negative is 1/50 while the probability that a person who does not have the disease tests positive is 1/10. John’s test returns positive.

1. Find the probability that John has the disease.

2. You are now told that this disease is hereditary. The probability that a son suffers from the disease if his father does is 4/5, the probability that a son is infected with the disease even though his father is not is 1/95. What is the probability that Steve has the disease given that his son Peter has the disease? (Note: You may assume that the disease only affects males so you can ignore the dependence on Peter’s mother’s health.)

Problem 3
A biased 4 sided die is rolled and the down face is a random variable \( N \) described by the following pmf:

\[
p_N(n) = \begin{cases} 
\frac{n}{15} & \text{for } n = 1,2,3,4 \\
0 & \text{otherwise.}
\end{cases}
\]

Given the random variable \( N \), a biased coin with bias \( \frac{N+1}{2N} \) is flipped and the random variable \( X \) is 1 or zero according to whether the coin shows heads or tails, i.e., the conditional pmf is

\[
p_{X|N}(x|n) = \left( \frac{n + 1}{2n} \right)^x \left( 1 - \frac{n + 1}{2n} \right)^{1-x} \quad \text{for } x = 0,1.
\]

1. Find the conditional pmf \( p_{N|X}(n|x) \).

2. Find the conditional expectation \( \mathbb{E}(N|X = 1) \).
Problem 4

(This problem uses the notion of ‘moment generating function’ that we introduced in class for its solution. It will not be required for the final.)

Let $X$ be a Gaussian random variable with $X \sim N(0, \sigma^2)$ and let $U$ be a Bernoulli random variable with $U \sim \text{Bernoulli}(\epsilon)$ independent of $X$. Define $V$ as

$$V = XU.$$ 

1. Find the moment generating function of $V$, $M_V(s) = \mathbb{E}(e^{sV})$.

2. Find the mean and variance of $V$. 