Probabilistic Systems Analysis

* An example of Joint pmf
* Conditional pmf and independence

I have a box with 3 types of balls

<table>
<thead>
<tr>
<th>Blue</th>
<th>Red</th>
<th>Green</th>
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<td>3</td>
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I draw 3 balls with replacement at random.

\[ \Omega = \{ (w_1, w_2, w_3): w_i \in \{B, R, G\} \} \]

\[ P(\{w\}) = \frac{1}{27} \quad (= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}) \]

\[ \omega = (w_1, w_2, w_3) \in \Omega \]

We are interested in the two random variables

\[ X(\omega) = \text{"number of blue balls among the ones drawn.} \]

\[ Y(\omega) = \text{"number of red balls.} \]

eg if \( \omega = (BGB) \)

then \( X(\omega) = 2 \), \( Y(\omega) = 0 \)

What is the pmf of \( X \)? \( X(\omega) \in \{0, 1, 2, 3\} \)

\[ p_X(k) = P(\{ \omega : X(\omega) = k \}) \]

\[ p_X(0) = \frac{1}{27} \cdot 3^3 = \frac{8}{27} \]

\[ p_X(1) = \frac{1}{27} \cdot 3 \cdot 2^2 = \frac{12}{27} \]

\[ p_X(2) = \frac{1}{27} \cdot 3 \cdot 1 = \frac{6}{27} \]

\[ p_X(3) = \frac{1}{27} \]
by symmetry

$P_X(0) = \frac{8}{27}$, $P_X(1) = \frac{12}{27}$, $P_X(2) = \frac{6}{27}$, $P_X(3) = \frac{1}{27}$

These numbers do not allow to answer the question

$P(\{\omega : X(\omega) \geq 1, Y(\omega) \leq 2\}) = ?$

joint pmf

$P_{XY}(k, e) = P(\{\omega : X(\omega) = k, Y(\omega) = e\})$

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tbody>
<tr>
<td>e</td>
<td></td>
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<tr>
<td>0</td>
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<td>3</td>
<td>$\frac{1}{27}$</td>
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General formula

$P_{XY}(k, e) = \binom{3}{k, e, 3-k-e} \frac{1}{3^3} \frac{1}{3^e} \frac{1}{3^{3-k-e}}$

binomial coefficient

$\binom{m}{k_1, k_2, k_3} = \frac{m!}{k_1! k_2! k_3!}$

- gives the number of sequences

BRB... GR

with $k_1$ B's, $k_2$ R's and $k_3$ G's

-
For $m$ balls with $n$ colors, this formula generalizes to
\[
\binom{m}{k_1, k_2, \ldots, k_n} = \frac{m!}{k_1! \cdot k_2! \cdot \ldots \cdot k_n!}
\]

For instance, suppose that I have a box, and the fraction of balls of type 1 is $p_1$, \ldots, fraction of type $n$ is $p_n\; p_1 + \ldots + p_n = 1$.

I extract $m$ balls with replacement.

$X_1 =$ "number of balls of type 1 among the $m$", $X_n =$ "number of type $n$".

Joint pmf
\[
P_{X_1 X_2 \ldots X_n}(k_1, k_2, \ldots, k_n)
\]
- if $k_1 + \ldots + k_n \neq m \Rightarrow p_{X_1 \ldots X_n}(k_1 \ldots k_n) = 0$
- otherwise
\[
P_{X_1 X_2 \ldots X_n}(k_1, k_2, \ldots, k_n) = \binom{m}{k_1, k_2, \ldots, k_n} \prod_{i=1}^{n} p_i^{k_i}
\]

(multinomial distribution, pmf).

Conditional pmf: $X, Y$ two r.v.'s
\[
P_{X|Y}(k|\omega) = P(\{\omega : X(\omega) = k \mid \{\omega : Y(\omega) = \omega \})
\]
By definition of conditional probability, we have

\[ P_{X|Y}(k|e) = \frac{P(\{\omega : X(\omega) = k, Y(\omega) = e\})}{P(\{\omega : Y(\omega) = e\})} \]

\[ P_{X|Y}(k|e) = \frac{P_{X,Y}(k,e)}{P_Y(e)} \]

Consider for instance the box with RGB balls.

\[ P_{X,Y}(k,e=0) = \frac{1}{27}, \frac{3}{27}, \frac{3}{27}, \frac{1}{27} \]

\[ P_Y(e=0) = \frac{8}{27} \]

Hence

\[ P_{X|Y}(k|e) = \begin{array}{cccc}
  k=0 & k=1 & k=2 & k=3 \\
  1/8 & 3/8 & 3/8 & 1/8 \\
\end{array} \]

\[ \mathbb{E} P_{X|Y}(k|e) = \left( \begin{array}{c}
  k=0 \\
  \binom{3}{k} \frac{1}{2^k} \frac{1}{2^{3-k}} \end{array} \right) \text{ why?} \]

We saw that two random variables \( X, Y \) are independent if \( \forall k, e \)

\[ P_{X,Y}(k,e) = P_X(k) P_Y(e) \]

By the definition of conditional pmf, this is equivalent to

\[ P_{X|Y}(k|e) = P_X(k) \]
Some standard calculation with joint pmfs.

- What is the probability that $X = Y$?

$$
P(\{\omega : X(\omega) = Y(\omega)\}) = \sum_k P(\{\omega : X(\omega) = k, Y(\omega) = k\})$$

$$= \sum_k p_{XY}(k,k)$$

If $X$ and $Y$ are independent

$$P(\{\omega : X = Y\}) = \sum_k p_X(k)p_Y(k).$$

- What is the probability that $\min(X,Y) = k$?

(assume $X,Y$ take integer values)

$$P(\{\omega : \min(X(\omega), Y(\omega)) = k\}) =$$

$$= \sum_{\ell, m > k} p_{XY}(\ell, m) - \sum_{\ell, m > k+1} p_{XY}(\ell, m)$$

Example: I throw two dice. What is the probability that the smallest is 3? $X,Y$ outcomes

$$p_{XY}(\ell, m) = \frac{1}{36} \text{ for } \ell, m \in \{1, \ldots, 6\}$$

$$P(\min(X,Y) = k) = \frac{1}{36} \cdot (6-k+1)^2 - \frac{1}{36} (6-k)^2$$

$$= \frac{12 - 2k + 1}{36}$$

$pmf$ of $Z = \min(X,Y)$

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<th>$k$</th>
<th>$p_{Z_k}$</th>
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