Solutions should be complete and concisely written. Please, mark clearly the beginning and end of each problem.

You have 3 hours. Try to solve as many problems as you can during this time, but keep in mind that you can also get a good grade by solving a subset of problems.

Points assigned to each problem are indicated in parenthesis. I recommend to look at all problems before starting.

For any clarification on the text, a TA will sit out of the room for part of the time, and I will be available in Packard 272. You can consult the Bertsekas and Tsitsiklis textbook, the reader and the lecture notes. You cannot consult other books, use computers, and in particular you cannot use the web.

Solutions should be written on the blue books. Please, write your name on each of the books.

Problem 1 (20 points)

You write a message that is $n = 1000$ characters long and transmit it through a noisy communication channel. The channel corrupts each entry independently with probability $p \in [0, 1]$. For instance (for brevity, this example has less than 1000 characters):

Input: Our doubts are traitors, and make us lose the good we oft might win, by fearing to attempt

Output: Our dou*ts are trai***s, and make us lose t*e good we oft m**ht win, by fearing to **tempt

(The original message does not contain the special erasure symbol * and whenever a character is corrupted, it gets replaced by *).

As you can see, isolated erasures are not very harmful, because most of the times the word can be guessed correctly even if one letter is missing. Define $N_2$ to be the number of sequences of two consecutive erasures (it is understood that these sequences can belong to larger groups of consecutive erasures). For instance, in the above example, $N_2 = 4$, because the pattern trai***s counts for two sequences of two consecutive erasure (the first and second, and the second and third erasures).

(a) Compute the expectation of this random variable, i.e. $\mathbb{E}\{N_2\}$.

(b) Use Markov inequality to derive an upper bound on the probability that there is many of these pairs. More precisely, find a quantity $m_M$ (depending on $p$) such that $\mathbb{P}\{N_2 \geq m_M\} \leq 1/10$.

(c) Compute the variance of $N_2$.

(d) Use Chebyshev inequality to improve the bound at point (b). As in point (b), you are requested to find a quantity $m_C$ (depending on $p$) such that $\mathbb{P}\{N_2 \geq m_C\} \leq 1/10$. However, using Chebyshev, you should be able to get $m_C < m_M$ (i.e. a tighter bound).

Problem 2 (15 points)

Consider the following model for packet arrival at an Internet router. Starting from time 0, the first packet arrives after $X_1$ time units, the second packet arrives $X_2$ time units after the first, and so on. In other
words, $X_k$ is the time between the $(k-1)$-th packet and the $k$-th one. The arrival time of the $k$-th packet is therefore
\[ T_k = X_1 + X_2 + \cdots + X_k. \quad (1) \]
We assume that the $X_i$’s are independent random variables, with exponential distribution. (Recall that this means that their common probability density function is $f_{X_i}(x) = e^{-x} \mathbb{1}_{x \geq 0}$.) We would like to show that the probability density function of $T_k$ is
\[ f_{T_k}(t) = \frac{t^{k-1}}{(k-1)!} e^{-t} \mathbb{1}_{t \geq 0}. \quad (2) \]

(a) Consider the case $k = 2$. Compute $f_{T_2}$ and compare it with Eq. (2).

(b) Assume the formula Eq. (2) is correct for $T_k$, and compute the probability density function of $T_{k+1}$. [Hint: It might be a good idea to compute the pdf of $T_{k+1}$ up to a normalization constant, and then use the fact that $\int_0^\infty t^m e^{-t} dt = m!$.]

(c) Let $N(t)$ denote the number of packet that arrive up to time $t$. Use the law of large numbers to show that
\[
\lim_{t \to \infty} \mathbb{P}(N(t) \leq 0.99 t) = 0, \quad (3)
\]
\[
\lim_{t \to \infty} \mathbb{P}(N(t) \geq 1.01 t) = 0. \quad (4)
\]

Problem 3 (15 points)

We are studying flower samples collected in a field. We know that the flowers belong to one of two species, and the two species differ because of the petal length:

Species 1: Mean petal length $\mu_1 = 3$ cm, standard deviation $\sigma_1 = 0.4$ cm.

Species 2: Mean petal length $\mu_2 = 4.5$ cm, standard deviation $\sigma_2 = 0.6$ cm.

We further know that the 80% of the flowers in the field are from species 1, and 20% from species 2, and that flowers are picked randomly.

(a) We assume that, given a species, the distribution of petal lengths is Gaussian, with the mean and standard deviation of that species.

We examine a new flower with petal length $x = 3.7$ cm. What is the (posterior) probability that this flower belongs to species 1 (to species 2)? Use the MAP rule to attribute the new flower to one of the two species.

(b) On the basis of some new theory, we change our model. Now the petal distributions are Laplace. This means that the probability density function for species $i$ is
\[ f_i(x) = \frac{1}{2a_i} \exp \left\{ - \frac{|x - b_i|}{a_i} \right\}. \quad (5) \]
Compute the parameters $a_i$, $b_i$ in terms of the mean $\mu_i$ and standard deviation $\sigma_i$. Sketch the two distributions.

(c) Repeat the exercise at point (a) for this new model. Namely, considering a new flower with petal length $x = 3.7$ cm, compute the (posterior) probability that this flower belongs to species 1 (to species 2).