HW4 Solutions

1. **(20 pts.) Mean and Variance**
Compute the mean and variance for the following problems in the previous HW:

(a) **(9 pts.)** Problem 1 of HW#3: The mean and variance of the number of packet losses in each of the three models.

**Answer:**
- **(3 pts.)** part (a): each packet is routed over a different path and is lost independently with probability $p$:
  
  $E[X] = np$
  
  $\text{Var}(X) = np(1 - p)$

- **(3 pts.)** part (b): all $n$ packets are routed along the same path, and with probability $p$ one of the links along the path fails and all $n$ packets are lost. Otherwise all packets are received. In this case $P(X = 0) = 1 - p$ and $P(X = n) = p$. Hence,
  
  $E[X] = 0 \times (1 - p) + n \times p = np.$
  
  $\text{Var}(X) = (0 - np)^2 \times (1 - p) + (n - np)^2 \times p = n^2p(1 - p)$

- **(3 pts.)** part (c): the $n$ packets are divided into 2 groups of $n/2$ packets, and each group is routed along a different path and lost with probability $p$. Losses of different groups are independent events. Let $Y_i$ represent whether path $i$ fails ($Y_i = 1$ if it fails, $Y_i = 0$ if it does not). Then $X = (n/2)(Y_1 + Y_2)$ and $Y_i \sim \text{Bern}(p)$. Hence, we get
  
  $E[X] = (n/2)E[Y_1 + Y_2] = (n/2)(E[Y_1] + E[Y_2]) = np$
  
  $\text{Var}(X) = (n/2)^2\text{Var}(Y_1 + Y_2) = (n/2)^2(\text{Var}(Y_1) + \text{Var}(Y_2)) = (n^2/2)p(1 - p)$

  where the last line is due to the independence of $Y_1, Y_2$.

(b) **(5 pts.)** Problem 2 of HW#3: The mean and variance of the number of Babe Ruth cards acquired.

**Answer:** If $X$ is the random variable denoting the number of cards Babe Ruth cards acquired, then $X \sim \text{Bin}(m, 1/n)$ and

$E[X] = \frac{m}{n}$

$\text{Var}(X) = \frac{m}{n} \left(1 - \frac{1}{n}\right)$

(c) **(6 pts.)** Problem 3 of HW#3: The mean and variance of $B$ and $G$ using a direct calculation.
Answer: As found in homework 3:

\[ \Pr(G = 0) = \frac{1}{25} \]
\[ \Pr(G = 1) = 1 - \frac{1}{25} \]

\[ \mathbb{E}[G] = \Pr(G = 0) \cdot 0 + \Pr(G = 1) \cdot 1 = \Pr(G = 1) = \frac{31}{32} \]

\[ \text{Var}(G) = \Pr(G = 0) \left( 0 - \frac{31}{32} \right)^2 + \Pr(G = 1) \left( 1 - \frac{31}{32} \right)^2 = \left( \frac{1}{32} \right)^2 + \left( \frac{31}{32} \right)^2 \left( \frac{1}{32} \right)^2 = \frac{31}{1024} \]

The distribution of \( B \) is as follows:

\[ \Pr(B = 0) = \frac{1}{2} \]
\[ \Pr(B = 1) = \frac{1}{2^2} \]
\[ \Pr(B = 2) = \frac{1}{2^3} \]
\[ \Pr(B = 3) = \frac{1}{2^4} \]
\[ \Pr(B = 4) = \frac{1}{2^5} \]
\[ \Pr(B = 5) = \frac{1}{2^5} \]

\[ \mathbb{E}[B] = \sum_{k=0}^{6} k \Pr(B = k) = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{32} = \frac{31}{32} \]

\[ \text{Var}(B) = \Pr(B = 0) \left( 0 - \frac{31}{32} \right)^2 + \Pr(B = 1) \left( 1 - \frac{31}{32} \right)^2 + \Pr(B = 2) \left( 2 - \frac{31}{32} \right)^2 + \Pr(B = 3) \left( 3 - \frac{31}{32} \right)^2 + \Pr(B = 4) \left( 4 - \frac{31}{32} \right)^2 + \Pr(B = 5) \left( 5 - \frac{31}{32} \right)^2 \]
\[ = \frac{1695}{1024} \]

Comment: Note that \( \mathbb{E}[B] = \mathbb{E}[G] \) above, which is a little counterintuitive. We might expect there to be more boys on average than girls, because the Browns might end up having many boys before their first girl. Or some might expect there to be more girls on average than boys, since the Browns will keep having children until they obtain a girl: thus they are guaranteed to have a girl, but there are no guarantees about boys. However, it turns out that the Browns’ strategy actually does not change the expected number of boys vs girls. In fact, it doesn’t matter when the family stops having children or what decision criteria they use; we will always have \( \mathbb{E}[B] = \mathbb{E}[G] \), as long as each birth is equally likely to be a boy or girl.
2. (20 pts.) Expectation
Al gambles against Gina. Each night Al draws a card from a deck (with replacement). If it is a spade or a queen, Al wins $4. If not, Al loses $1. Let $X$ be the random variable corresponding to Al’s total winnings after 30 nights.

(a) (10 pts.) Specify the pmf of the random variable $X$.
(b) (10 pts.) Find its mean.

Answer:

(a) Note that $X = 5W - 30$, where denote the number of times Al wins by $W$. $W$ is a binomial random variable with parameters $n = 30$ and $p = \frac{4}{13}$, since

\[ P\{\text{Spade or Queen}\} = P\{\text{Spade}\} + P\{\text{Queen}\} - P\{\text{Spade and Queen}\} = \frac{16}{52} = \frac{4}{13}. \]

As a result,

\[ p_X(x) = P(X = x) = P(5W - 30 = x) = P\left(W = \frac{x}{5} + 6\right) = \left(\frac{30}{\frac{x}{5} + 6}\right) \left(\frac{4}{13}\right)^{\left(\frac{x}{5} + 6\right)} \left(\frac{9}{13}\right)^{(24 - \frac{x}{5})} \]

if $x = 5w - 30$ for an integer $w$ between 0 and 30 and zero otherwise.

(b) Let $\chi$ denote the set of possible values of $x$,

\[ \mathbb{E}(X) = \sum_{x \in \chi} x p_X(x) = \sum_{x \in \chi} x P(X = x) = \sum_{w=1}^{30} (5w - 30) P(W = w) = 5 \sum_{w=1}^{30} w P(W = w) - 30 \sum_{w=1}^{30} P(W = w) = 5 \cdot 30 \cdot \frac{4}{13} - 30 = \frac{210}{13}, \]

where the last step follows from the fact that $\sum_{w=1}^{30} P_W(w) = 1$ (the pmf must add up to one over its domain) and that the mean of a binomial random variable of parameters $n$ and $p$ is equal to $np$. The result can also be derived using linearity of expectation.

3. (20 pts.) Defenestration
A famous ubiquitous operating system called Defenestration is run on a lot of computers. For each of the mutually independent computers, the probability mass function for $X$, the number of operating system crashes in a day, is given by

\[ p_X(k) = \frac{4 - k}{10}; \quad k = 0, 1, 2, 3. \]
On a day when for a given computer the operating system crashes $X = k$ times, the user has a probability of $1 - 2^{-k}$ of reinstalling the operating system.

(a) **(6 pts.)** Find the mean and variance of $X$.

(b) **(7 pts.)** Find the probability that a particular computer has its operating system reinstalled on a given day.

(c) **(7 pts.)** In a given group of 10 computers, what is the probability that exactly three of them had their operating systems reinstalled on a particular day?

**Answer:**

(a) The mean of $X$ is

$$E[X] = \left( \frac{4 - 0}{10} \right) \times 0 + \left( \frac{4 - 1}{10} \right) \times 1 + \left( \frac{4 - 2}{10} \right) \times 2 + \left( \frac{4 - 3}{10} \right) \times 3 = 1.$$  

The second moment can be calculated as

$$E[X^2] = \left( \frac{4 - 0}{10} \right) \times 0 + \left( \frac{4 - 1}{10} \right) \times 1 + \left( \frac{4 - 2}{10} \right) \times 4 + \left( \frac{4 - 3}{10} \right) \times 9 = 2.$$  

Thus,

$$\text{Var}(X) = E[X^2] - (E[X])^2 = 1.$$ 

(b) Let $R$ denote the event that a particular computer has its operating system reinstalled. Using total probability,

$$P(R) = \sum_{k=0}^{3} P(R | X = k) p_X(k)$$

$$= \sum_{k=0}^{3} (1 - 2^{-k}) \left( \frac{4 - k}{10} \right)$$

$$= \frac{3}{10} \times \frac{1}{2} + \frac{2}{10} \times \frac{3}{4} + \frac{1}{10} \times \frac{7}{8}$$

$$= \frac{31}{80} = 0.3875.$$ 

(c) There are $\binom{10}{3}$ ways to have three computers out of 10 have their OS reinstalled and each has probability $p = P(R) = 31/80$, thus the answer is

$$\binom{10}{3} \left( \frac{31}{80} \right)^3 \left( \frac{49}{80} \right)^7.$$ 
