Homework #4 Solutions

1. **Mixed random variable.**
   a. For \( x < 0 \), the cdf is 0, and for \( x > 2 \) the cdf is 1. For values of \( x \) between 0 and 2, we can find the cdf using the law of total probability. We can assume without loss of generality that Alice bets on heads. Then
   \[
   F_X(x) = \Pr \{ X \leq x \} = \Pr \{ X \leq x, \ H \} + \Pr \{ X \leq x, \ T \} 
   = \frac{1}{2} (\Pr \{ X \leq x \mid H \} + \Pr \{ X \leq x \mid T \}) 
   = \frac{1}{2} (\Pr \{ U \leq x - 1 \} + \Pr \{ U \geq 1 - x \}) 
   = \begin{cases} 
   \frac{1}{2} (1 - (1 - x)) & 0 \leq x \leq 1 \\
   \frac{1}{2} + \frac{1}{2} (x - 1) & 1 < x \leq 2 
   \end{cases} 
   = \frac{x}{2} \text{ for } 0 \leq x \leq 2.
   
   Thus \( X \sim U[0, 2] \), that is, \( X \) is uniformly distributed between 0 and 2.

2. **Probabilities from CDF.**
   a. \( \Pr \{ X = 2 \} = F_X(2) - F_X(2^-) = 2/3 - 1/3 = 1/3. \)
   b. \( \Pr \{ X < 2 \} = F_X(2^-) = 1/3. \)
   c. From the figure
   \[
   \Pr \{ \{ X = 2 \} \cup \{ 0.5 \leq X \leq 1.5 \} \} = \Pr \{ X = 2 \} + F_X(1.5) - F_X(0.5) 
   = \frac{1}{3} + \frac{1}{3} - \frac{1}{3} (0.5)^2 
   = \frac{1}{3} + \frac{1}{3} - \frac{1}{12} = \frac{7}{12}.
   
   d. From the figure
   \[
   \Pr \{ \{ X = 2 \} \cup \{ 0.5 \leq X \leq 3 \} \} = \Pr \{ 0.5 \leq X \leq 3 \} 
   = F_X(3) - F_X(0.5) 
   = \frac{5}{6} - \frac{1}{3} = \frac{3}{4}.
   
3. **Quantizer.**
a. If $k < 0$, then $p_Y(k) = 0$. If $k \geq 0$, then
\[
p_Y(k) = P\{Y = k\} = P\{k \leq X < k + 1\} = F_X(k+1) - F_X(k)
= (1 - e^{-\lambda(k+1)}) - (1 - e^{-\lambda k})
= e^{-\lambda k} - e^{-\lambda(k+1)} = e^{-\lambda k}(1 - e^{-\lambda}).
\]
In other words, $Y$, the quantization of exponential random variable $X$, is $\text{Geom}(e^{-\lambda})$, where $e^{-\lambda}$ is the probability of success in one time unit.

b. Since for a continuous r.v., the probability of any given point is 0,
\[
P\{X < x\} = P\{X \leq x\} = F_X(x).
\]
Now, since
\[
0 \leq Z = X - Y = X - \lfloor X \rfloor < 1,
\]
the quantization error pdf $f_Z(z)$ is 0 if $z < 0$ or $z \geq 1$.
If $0 \leq z < 1$, then
\[
F_Z(z) = P\{Z \leq z\} = P(0 \leq Z \leq z)
= \sum_{k=0}^{\infty} P\{k \leq X \leq k + z\}
= (1 - e^{-\lambda z}) \sum_{k=0}^{\infty} e^{-\lambda k}
= \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}.
\]
Differentiating with respect to $z$, we obtain
\[
f_Z(z) = \frac{\lambda e^{-\lambda z}}{1 - e^{-\lambda}}, \quad 0 \leq z < 1.
\]


a. The mean of $X$ is
\[
E(X) = \left(\frac{4 - 0}{10}\right) \times 0 + \left(\frac{4 - 1}{10}\right) \times 1 + \left(\frac{4 - 2}{10}\right) \times 2 + \left(\frac{4 - 3}{10}\right) \times 3 = 1.
\]
The second moment can be calculated as
\[
E(X^2) = \left(\frac{4 - 0}{10}\right) \times 0 + \left(\frac{4 - 1}{10}\right) \times 1 + \left(\frac{4 - 2}{10}\right) \times 4 + \left(\frac{4 - 3}{10}\right) \times 9 = 2.
\]
Thus,
\[
\sigma_X^2 = E(X^2) - (E(X))^2 = 1.
\]
b. Let $R$ denote the event that a particular computer has its operating system reinstalled. Using total probability,
\[
P(R) = \sum_{k=0}^{3} P(R|X = k)p_X(k)
\]
\[
= \sum_{k=0}^{3} (1 - 2^{-k}) \left( \frac{4 - k}{10} \right)
\]
\[
= \frac{3}{10} \times \frac{1}{2} + \frac{2}{10} \times \frac{3}{4} + \frac{1}{10} \times \frac{7}{8}
\]
\[
= \frac{31}{80} = 0.3875.
\]

c. There are $\binom{10}{3}$ ways to have three computers out of 10 have their OS reinstalled and each has probability $p = P(R) = \frac{31}{80}$, thus the answer is
\[
\binom{10}{3} \left( \frac{31}{80} \right)^3 \left( \frac{49}{80} \right)^7.
\]

5. **Geometric random variable.**

    clear all;clc;close all;

    N = 100000;
    G = ones(N,1);
    p = 0.3;  % p = 0.3

    for i=1:N
        while rand>p
            G(i) = G(i)+1;  % Fliping a coin
        end
    end

    hist_G = hist(G,1:35);  % histogram of G
    pmf_G = p*(1-p).^0:34;  % pmf of G

    I3 = length(find(G>3));  % number of G>3

    plot(1:35,hist_G/N,'rs',1:35,pmf_G,...
         'bo',4:35,hist_G(4:35)/I3,'kx');  % Draw a histogram

    a. Can see the histogram and the pmf coincides.

    b. Conditional histogram is shifted version of the original histogram. This shows the memoryless property of geometric random variable.