Homework #7 Solutions

1. Uncorrelation vs. independence.
   
   a. Since $A$ and $\Theta$ are independent,
      
      \[
      \begin{align*}
      E(X) &= E(A \cos \Theta) = E(A) E(\cos \Theta) = 0 \cdot 0 = 0, \Var(X) \\
      &= E(X^2) - (E(X))^2 = E(A^2 \cos^2 \Theta) - 0 \\
      &= E(A^2) E(\cos^2 \Theta) = E(\cos^2 \Theta) = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 \theta \, d\theta = \frac{1}{2}.
      \end{align*}
      \]
      
   b. To be uncorrelated, $E(XY)$ must equal $E(X) E(Y)$.
      
      \[
      E(XY) = E(A \cos \Theta \cdot A \sin \Theta) = E(A^2) E(\cos \Theta \cdot \sin \Theta) = E(0.5 \sin 2\Theta) = 0.
      \]
      Therefore $X$ and $Y$ are uncorrelated.
      
   c. It is much easier to disprove independence than to prove it; a disproof requires only a single counterexample, while a proof of independence requires one to find the pdfs, which can be a daunting task. One way to create a counterexample is to find two functions such that $E(g(X)h(Y)) \neq E(g(X))E(h(Y))$. Let $g(x) = h(x) = x^2$. Then
      
      \[
      E(X^2 Y^2) = \frac{1}{8} E(A^4) = \frac{3}{8} \neq E(X^2) E(Y^2) = \frac{1}{4}.
      \]
      Here we used the fact that the fourth moment of $A \sim N(0,1)$ is $E(A^4) = 3$.

2. Balls and Bins. Define the indicator random variable
      
      \[
      Y_i = \begin{cases} 
      1 & \text{if bin } i \text{ is empty} \\
      0 & \text{otherwise}.
      \end{cases}
      \]
      Then $X = \sum_{i=1}^{50} Y_i$ and
      
      \[
      E(X) = \sum_{i=1}^{50} E(Y_i) = 50 \times E(Y_1).
      \]
      But
      
      \[
      E(Y_1) = P\{\text{bin 1 is empty}\} = \prod_{i=1}^{100} P\{\text{ball } i \text{ does not fall in bin 1}\}
      \]
      
      \[
      = (P\{\text{ball 1 does not fall in bin 1}\})^{100} = \left(\frac{49}{50}\right)^{100} = 0.13262.
      \]
      Thus
      
      \[
      E(X) = 6.631.
      \]
3. Correlation coefficients.

\[
\text{Cov}(A, B) = \text{E}[AB] - \text{E}[A]\text{E}[B] \\
= \text{E}[WX + WY + X^2 + XY] - 0 \\
= \text{E}[X^2] \\
= 1.
\]

Also,

\[
\text{Var}[A] = \text{Var}[X + Y] \\
= \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}(X, Y) \\
= 2.
\]

Similarly, \(\text{Var}[B] = 2\). Therefore,

\[
\rho_{A,B} = \frac{\text{Cov}(A, B)}{\sqrt{\text{Var}[A]\text{Var}[B]}} = \frac{1}{2}.
\]

On the other hand,

\[
\text{Cov}(A, C) = \text{E}[AC] - \text{E}[A]\text{E}[C] \\
= \text{E}[WX + XY + WZ + XZ] - 0 \\
= 0.
\]

Therefore,

\[
\rho_{A,C} = \frac{\text{Cov}(A, C)}{\sqrt{\text{Var}[A]\text{Var}[C]}} = 0.
\]

4. Modified additive noise channel. First we find the mean and variance of \(Y\) and its covariance with \(X\):

\[
\text{E}(Y) = \text{E}(abX + bZ) = ab\text{E}(X) + b\text{E}(Z) = 0. \\
\text{Var}(Y) = \text{E}(abX + bZ)^2 - (\text{E}(abX + bZ))^2 \\
= a^2b^2\text{E}(X^2) + 2ab^2\text{E}(XE) + b^2\text{E}(Z^2) - \text{E}(Y) = a^2b^2P + 0 + b^2N - 0 \\
= a^2b^2P + b^2N.
\]

\[
\text{Cov}(X, Y) = \text{E}[(X - \text{E}X)(Y - \text{E}Y)] = \text{E}(XY) \\
= \text{E}[X(abX + bZ)] = ab\text{E}X^2 + b\text{E}XZ \\
= abP + b\text{E}XZ = abP.
\]

The MMSE linear estimate of \(X\) given \(Y\) is given by

\[
\hat{X} = \frac{\text{Cov}(X, Y)}{\sigma_Y^2} (Y - \text{E}Y) + \text{E}X = \frac{abP}{a^2b^2P + b^2N} (Y - \text{E}Y) + \text{E}X = \frac{aP}{b(a^2P + N)} Y.
\]

The MSE of the linear estimate is the minimum MMSE:

\[
\text{MMSE} = \sigma_X^2 - \frac{\text{Cov}^2(X, Y)}{\sigma_Y^2} = P - \frac{a^2b^2P^2}{a^2b^2P + b^2N} = \frac{a^2P^2 + PN - a^2P^2}{a^2P + N} = \frac{PN}{a^2P + N}.
\]
5. *Randomly biased coin.*

a. We have

\[
E[X_N] = \sum_{i=1}^{N} E[Y_i] = \frac{N}{2}
\]

\[
E[U] = \frac{1}{2}
\]

\[
E[X_N^2] = \sum_{i=1}^{N} E[Y_i^2] + \sum_{i=1}^{N} \sum_{j\neq i} E[Y_iY_j]
\]

\[
= \sum_{i=1}^{N} E[Y_i] + \sum_{i=1}^{N} \sum_{j\neq i} E[Y_iY_j]
\]

\[
= \frac{N}{2} + \sum_{i=1}^{N} \sum_{j\neq i} \frac{1}{3}
\]

\[
= \frac{N}{2} + \frac{N(N-1)}{3}
\]

\[
= \frac{2N^2 + N}{6}
\]

\[
E[U^2] = \frac{1}{3}
\]

\[
\text{Var}[X_N] = \frac{2N^2 + N}{6} - \frac{N^2}{4} = \frac{N^2 + 2N}{12}
\]

Also,

\[
E[X_NU] = \sum_{i=1}^{N} E[Y_iU]
\]

\[
= NE[Y_1U]
\]

\[
= N \int_{0}^{1} u^2 \, du
\]

\[
= \frac{N}{3}.
\]

Therefore,

\[
\text{Cov}(X_N, U) = \frac{N}{3} - \frac{N}{4} = \frac{N}{12}
\]
The linear MMSE estimator of $U$ given $X_N$ is given by

$$
\hat{U} = \frac{\text{Cov}(X_N, U)}{\sigma^2_{X_N}} (X_N - \mathbb{E}[X_N]) + \mathbb{E}[U]
$$

$$
= \frac{N}{N^2 + 2N} (X_N - \frac{N}{2}) + \frac{1}{2}
$$

$$
= \frac{1}{N+2} X_N + \frac{1}{N+2}.
$$

b.

clear all;clc;close all;
u = 0.84; % Bias
N_max = 1000; % Total sample number
samples = rand(N_max,1)<u; % Bernoulli(u)
U_hat = zeros(1,N_max-1);

for N=2:N_max
    % count the number of head = x_N
    x_N = sum(samples(1:N));

    % Compute U_hat
    U_hat(N-1) = 1/(N+2) * x_N + 1/(N+2);
end

% plot
plot(2:N_max, U_hat,'b',2:N_max, ones(1,N_max-1)*u,'r');