EE178: Midterm Solutions

1. Rolling Dice (50 points)
   You roll a fair die two times. Consider the following events:
   
   \[ A = \text{the sum of the first and second roll is 5} \]
   \[ B = \text{the sum of the first and second roll is 7} \]
   \[ C = \text{the first roll is different from the second roll} \]

   Let \( X \) be the random variable representing the outcome of the first roll.

   (a) Is event \( A \) independent of the event \( C \)? (7 points)
   (b) Is event \( B \) independent of the event \( C \)? (7 points)
   (c) Find the conditional pmf of \( X \) given \( A \), i.e., find \( P(X = k | A) \) for \( 1 \leq k \leq 6 \). (6 points)
   (d) Find the conditional pmf of \( X \) given \( C \). (6 points)

Now, you roll a fair die four times. Let \( Y \) be a random variable corresponding to the sum of the four rolls.

   (e) Find the size of the set \( \{ \omega : Y(\omega) = 6 \} \). In other words, how many possible ways to get \( Y = 6 \)? (6 points)
   (f) What is the probability of \( Y = 6 \)? (6 points)

Now, you roll a fair twenty-sided die 10 times. Let \( Z \) be a random variable corresponding to the sum of the ten rolls.

   (g) Find the size of the set \( \{ \omega : Z(\omega) = 23 \} \). In other words, how many possible ways to get \( Z = 23 \)? (6 points)
   (h) What is the probability of \( Z = 23 \)? (6 points)

Solution:

(a) \( P(A) = \frac{4}{36} \) and \( P(C) = \frac{30}{36} \) where \( P(A \cap C) = \frac{4}{36} \). They are not independent.
(b) \( P(B) = \frac{6}{36} \) and \( P(C) = \frac{30}{36} \) where \( P(A \cap C) = \frac{6}{36} \). They are not independent.
(c) For \( 1 \leq k \leq 4 \),
   
   \[ P(X = k | A) = \frac{P(\{X = k\} \cap A)}{P(A)} \]
\[
\frac{P(\{(k, 5 - k)\})}{P(A)} = \frac{1}{4}
\]

For \( k = 5, 6 \), it is not hard to show that \( P(X = k|A) = 0 \).

(d) For \( 1 \leq k \leq 6 \),

\[
P(X = k|A) = \frac{P(\{X = k\} \cap A)}{P(A)}
= \frac{P(\{((k, 7 - k)\})}{P(A)}
= \frac{1}{6}
\]

(e) The size of the set \( \{\omega : Y(\omega) = 6\} \) is equal to the number of positive integer solutions of \( x_1 + x_2 + x_3 + x_4 = 6 \). We have seen that the number of solutions are \( \binom{5}{3} = 10 \).

(f) Since \( P(\omega) = \frac{1}{6^4} \) for all \( \omega \in \Omega \), the probability is \( P(Y = 9) = \frac{10}{6^4} \).

(g) The size of the set \( \{\omega : Z(\omega) = 23\} \) is equal to the number of positive integer solutions of \( x_1 + x_2 + \cdots + x_{10} = 23 \). We have seen that the number of solutions are \( \binom{22}{9} \).

(h) Since \( P(\omega) = \frac{1}{20^5} \) for all \( \omega \in \Omega \), the probability is \( P(Z = 9) = \frac{\binom{22}{9}}{20^5} \).
2. Batteries. (50 points)
There are two types of batteries in a bin: type 1 and type 2. When in use, type \( i \) batteries last (in hours) an exponentially distributed time with rate \( \lambda_i \), where \( \lambda_1 > \lambda_2 > 0 \). A battery that is randomly chosen from the bin will be a type \( i \) battery with positive probability \( p_i \), where \( p_1 + p_2 = 1 \). Let \( X \) be a random variable indicating the life time of a randomly chosen battery.

(a) Find the probability of the chosen battery having life time smaller than 5 given that the type 1 battery is chosen. I.e., what is \( P(X \leq 5|\text{type 1 battery is chosen}) \)? (8 points)

(b) Find the probability of the chosen battery having life time smaller than 5. I.e., what is \( P(X \leq 5) \)? (7 points)

(c) Find the cdf of \( X \). (7 points)

(d) Find the pdf of \( X \). Is \( X \) an exponential random variable? (7 points)

(e) Find the expected value of \( X \). (7 points)

(f) Find the variance of \( X \). (7 points)

(g) If a randomly chosen battery is still operating after \( t \) hours of use, what is the probability that it will still be operating after an additional \( s \) hours? (7 points)

Hint: You can use the following results without proof.

\[
\int_0^\infty x \cdot e^{-ax} \, dx = \frac{1}{a^2} \\
\int_0^\infty x^2 \cdot e^{-ax} \, dx = \frac{2}{a^3}.
\]

Solution:

(a) \[
P(X \leq 5|\text{type 1 battery is chosen}) = \int_0^5 \lambda_1 e^{-\lambda_1 x} \, dx = 1 - e^{-5\lambda_1}.
\]

(b) \[
P(X \leq 5) = p_1 \int_0^5 \lambda_1 e^{-\lambda_1 x} \, dx + p_2 \int_0^5 \lambda_2 e^{-\lambda_2 x} \, dx \\
= p_1 (1 - e^{-5\lambda_1}) + p_2 (1 - e^{-5\lambda_2}).
\]
(c) 
\[ P(X \leq x) = P(\text{type 1 is chosen})P(X \leq x|\text{type 1 is chosen}) \]
\[ + P(\text{type 2 is chosen})P(X \leq x|\text{type 2 is chosen}) \]
\[ = p_1 (1 - e^{-\lambda_1 x}) + p_2 (1 - e^{-\lambda_2 x}) \]
\[ = 1 - p_1 e^{-\lambda_1 x} - p_2 e^{-\lambda_2 x}. \]

(d) 
\[ f_X(x) = \frac{d}{dx} \{ p_1 (1 - e^{-\lambda_1 x}) + p_2 (1 - e^{-\lambda_2 x}) \} \]
\[ = p_1 \lambda_1 e^{-\lambda_1 x} + p_2 \lambda_2 e^{-\lambda_2 x} \]

It is NOT an exponential random variable.

(e) 
\[ \mathbb{E}[X] = \int_0^\infty x \left( p_1 \lambda_1 e^{-\lambda_1 x} + p_2 \lambda_2 e^{-\lambda_2 x} \right) dx \]
\[ = p_1 \int_0^\infty x \lambda_1 e^{-\lambda_1 x} dx + p_2 \int_0^\infty x \lambda_2 e^{-\lambda_2 x} dx \]
\[ = \frac{p_1}{\lambda_1} + \frac{p_2}{\lambda_2}. \]

(f) 
\[ \mathbb{E}[X^2] = \int_0^\infty x^2 \left( p_1 \lambda_1 e^{-\lambda_1 x} + p_2 \lambda_2 e^{-\lambda_2 x} \right) dx \]
\[ = p_1 \int_0^\infty x^2 \lambda_1 e^{-\lambda_1 x} dx + p_2 \int_0^\infty x^2 \lambda_2 e^{-\lambda_2 x} dx \]
\[ = \frac{2p_1}{\lambda_1^2} + \frac{2p_2}{\lambda_2^2}. \]

Therefore, 
\[ \text{Var}[X] = \frac{2p_1}{\lambda_1^2} + \frac{2p_2}{\lambda_2^2} - \left( \frac{p_1}{\lambda_1} + \frac{p_2}{\lambda_2} \right)^2. \]

(g) 
\[ P(X > t + s|X > t) = \frac{P(X > t + s)}{P(X > t)} \]
\[ = \frac{P(X > t + s)}{P(X > t)} \]
\[ = \frac{p_1 e^{-\lambda_1(t+s)} + p_2 e^{-\lambda_2(t+s)}}{p_1 e^{-\lambda_1 t} + p_2 e^{-\lambda_2 t}}. \]

Note that \( X \) does not have a memoryless property.
3. **M&M. (20 points)**

The blue M&M was introduced in 1995. Before then, the color mix in a bag of plain M&Ms was (30% Brown, 20% Yellow, 20% Red, 10% Green, 10% Orange, 10% Tan). Afterward it was (24% Blue, 20% Green, 16% Orange, 14% Yellow, 13% Red, 13% Brown). A friend of mine has two bags of M&Ms, and he tells me that one is from 1994 and one from 1996. He won’t tell me which is which, but he gives me one M&M from each bag. One is yellow and one is green. What is the probability that the yellow M&M came from the 1994 bag?

**Solution:**

Let $A$ be an event that Bag #1 from 1994 and Bag #2 from 1996, and $B$ be an event that Bag #1 from 1996 and Bag #2 from 1994. We have $P(A) = P(B) = \frac{1}{2}$. What you observed is the event $E$ that yellow from Bag #1 and green from Bag #2. Therefore,

\[
P(A|E) = \frac{P(E|A)P(A)}{P(E)} = \frac{0.2 \times 0.2 \times \frac{1}{2}}{0.2 \times 0.2 \times \frac{1}{2} + 0.1 \times 0.14 \times \frac{1}{2}} \approx 0.74.
\]