1. **DSB-SC modulator** (Lathi & Ding 4.2-3). You are asked to design a DSB-SC modulator to generate a modulated signal $km(t) \cos(\omega_c t + \theta)$, where $m(t)$ is a signal bandlimited to $B$ Hz. Figure P4.2-3 shows a DSB-SC modulator available in the stockroom. The carrier generator available generates not $\cos \omega_c t$ but $\cos^3 \omega_c t$. Explain whether you would be able to generate the design using only this equipment. You may use any kind of filter you like.

- **a.** What kind of filter is required in Fig. P4.2.3?
- **b.** Determine the signal spectra at points $b$ and $c$, and indicate the frequency bands occupied by these spectra.
- **c.** What is the minimum usable value of $\omega_c$?
- **d.** Would this scheme work in the carrier generator output were $\sin^3 \omega_c t$? Explain.
- **e.** Would this scheme work in the carrier generator output were $\cos^n \omega_c t$ for any integer $n \geq 2$?

**Solution (10 points)**

**a.** Using trigonometric identities or Euler’s formula, we can expand $\cos^3$ as

$$
\cos^3 \omega_c t = \frac{3}{4} \cos \omega_c t + \frac{1}{4} \cos 3\omega_c t.
$$

When modulated by $\cos^3 \omega_c t$, the transmitted signal contains the term $\frac{3}{4} m(t) \cos \omega_c t$, which is the desired modulated signal with spectrum centered at $f_c$. The other term has spectrum centered at $3f_c$. A bandpass filter centered at $\pm f_c$ allows the passage of the desired term but suppresses the unwanted term.

**b.** The signal spectra at points $b$ and $c$ are shown in the following figure.

![Signal Spectra](image)

**c.** In order to avoid spectral folding overlap after modulation, the minimum usable value of $f_c$ is $B$, where $B$ is the bandwidth of the lowpass signal $m(t)$.

**d.** Yes. Since $\sin^3 \omega_c t$ is $\cos^3 \omega_c t$ delayed by $\pi/2$, if we modulate with $\sin^3 \omega_c t = \cos^3(\omega_c t - \pi/2)$, then we can demodulate with $\cos^3(\omega_c t - \pi/2) = \sin^3 \omega_c t$. 
e. The expansion of \( \cos^n \omega_c t \) contains a term \( a_1 \cos \omega_c t \) when \( n \) is odd but not when \( n \) is even. Therefore the system works for carrier \( \cos^n \omega_c t \) if and only if \( n \) is odd.

2. *Audio scrambler* (Lathi & Ding 4.2-8). The system shown in Fig. P4.2-8 is used for scrambling audio signals. The output \( y(t) \) is the scrambled version of the input \( m(t) \).

![Audio scrambler diagram]

a. Find the spectrum of the scrambled signal \( y(t) \).

b. Suggest a method for descrambling \( y(t) \) to obtain \( m(t) \).

**SOLUTION** (10 points)

a. The spectrum \( Y(f) \) of the scrambled signal is shown in the following figure.

![Spectrum diagram]

b. Observe that \( Y(f) \) is the same as \( M(f) \) with frequency spectrum inverted, that is, the high frequencies are shifted to lower frequencies and vice versa. To get back to the original spectrum, we must invert \( Y(f) \), which can be done using the same scrambler.

3. *AM signal* (Lathi & Ding 4.3-1). In an amplitude modulation system, the message signal is given by Fig. P4.3-1 and the carrier frequency is 1 KHz. The modulator output is

\[
s_{AM}(t) = 2(b + 0.5m(t)) \cos \omega_c t.
\]

a. Determine the average message power.

b. If \( b = 1 \), determine the modulation index and the modulation power efficiency.

c. Sketch the modulated signal of part (a) in the time domain.

d. If \( b = 0.5 \), repeat parts (a) and (b).

**SOLUTION** (10 points)

a. In general, the sawtooth signal with period \( T \) and amplitude \( a \) is defined by its values in one period, \([-T/2, T/2] \):

\[
m(t) = \frac{2a}{T} t, \quad T/2 \leq t \leq T/2
\]
The power of this signal is

\[
m^2(t) = \frac{1}{T} \int_{-T/2}^{T/2} m^2(t) \, dt = \frac{2}{T} \int_{0}^{T/2} m^2(t) \, dt
\]

\[
= \frac{2}{T} \int_{0}^{T/2} \left(\frac{2a}{T}t\right)^2 \, dt = \frac{8a^2}{T^3} \int_{0}^{T/2} t^2 \, dt = \frac{8a^2}{T^3} \left[ \frac{T^2}{3} \right]_0^{T/2} = \frac{8a^2}{T^3} \frac{T^3}{3} \cdot 8 = \frac{a^2}{3}.
\]

Note that the power is independent of \(T\) and is proportional to the square of the amplitude. For the sawtooth signal of this problem, \(a = 2\) and \(T = 0.1\), so the power is

\[
P_m = \frac{2^2}{3} = \frac{4}{3}.
\]

We can rewrite \(s_{AM}(t)\) as

\[
s_{AM}(t) = (2b + m(t)) \cos \omega_c t.
\]

Therefore the power of the modulated signal is

\[
P_s = P_c + P_m = \frac{1}{2} (2b)^2 + \frac{1}{2} \left(\frac{4}{3}\right) = 2b^2 + \frac{2}{3}.
\]

b. The peak value of \(m(t)\) is 2. If \(b = 1\), the modulation index is \(m_p/A = 2/2b = 1\). The modulation power efficiency is

\[
\eta = \frac{P_s}{P_c + P_s} = \frac{2/3}{2 + 2/3} = \frac{1}{4} = 25\%.
\]

c. Shown below is the modulated output for one cycle of the input signal.

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\[
\begin{array}{c}
\text{d. If } b = 0.5, \text{ the modulation index is } m_p/A = 2/2b = 2. \text{ The modulation power efficiency is}
\end{array}
\]

\[
\eta = \frac{P_s}{P_c + P_s} = \frac{2/3}{1/2 + 2/3} = \frac{4}{7} = 57\%.
\]

e. Shown below is the modulated output for one cycle of the input signal.
Since $\mu > 1$, the input signal cannot be recovered by envelope detection.

4. **Equalization in AM modulation.** Consider the DSB-SC system shown in the figure below, where $s(t) = m(t) \cos 2\pi f_c t$ with $f_c \gg B$.

Assume that the noise $n(t)$ has power spectral density $S_n(f) = 0.1 \text{ mW/Hz}$. The PSD $S_m(f)$ of $m(t)$ in mW/Hz is

$$S_m(f) = \begin{cases} 10 - 10|f|/B & |f| \leq B \\ 0 & |f| > B. \end{cases}$$

The frequency response of the channel $H(f)$ is

$$H(f) = \begin{cases} 10 & |f - f_c| \leq B/2 \\ 0.5 & B/2 < |f - f_c| < B \\ 0 & \text{otherwise} \end{cases}$$

a. What is the power of $m(t)$?

b. Sketch the PSD of the modulated signal $s(t)$.

c. Find the equalizer $H_{eq}(f)$ such that in the absence of noise (i.e., for $n(t) = 0$), $z(t) = m(t)$.

d. Find the PSD and power of $z(t)$ due to noise only (i.e., for $m(t) = 0$), and due to signal only (i.e., for $n(t) = 0$), using $H_{eq}(f)$ is the equalizer you found in part (c).

e. Find the SNR of the receiver output with the equalizer of part (c).

**SOLUTION (25 points)**

a. The power spectral density of $m(t)$ is shown in the following figure.

The total power in mW is the area beneath the triangle:

$$P_m = \int_{-\infty}^{\infty} S_m(f) \, df = 10B .$$

b. By the modulation theorem, the PSD of the modulated signal is

$$S_s(f) = \frac{1}{2} \left( S_m(f + f_c) + S_m(f - f_c) \right) .$$
c. The required equalizer is $H_{eq}(f) = 2/H(f)$. The factor of 2 is needed because $\cos 2\pi f_c t$ in the demodulator reduces the signal amplitude by a factor of 2.

$$H_{eq}(f) = \begin{cases} 0.2 & |f - f_c| \leq B/2 \\ 4 & B/2 < |f - f_c| < B \\ 0 & \text{otherwise} \end{cases}$$

d. If the signal is not present, all of the output is due to the noise, so the output PSD is

$$S_{zn}(f) = 0.1 \cdot \frac{1}{B} \left( |H_{eq}(f + f_c)|^2 + |H_{eq}(f - f_c)|^2 \right) \cdot |H_{LPF}(f)|^2.$$ 

The PSD of $z(t)$ due to noise only and due to signal only are shown in the following figure.

Thus the total noise power is

$$P_n = \int_{-\infty}^{\infty} S_{zn}(f) \, df = 0.8 \cdot B + 0.002 \cdot B = 0.802B \text{ mW}.$$ 

e. From part (d),

$$\text{SNR} = \frac{P_m}{P_n} = \frac{10B}{0.802B} = 12.47 = 11.0 \text{ dB}.$$ 

5. **Frequency and phase offset in QAM system.** In a QAM system (Fig. 4.19 in the textbook), the locally generated carrier has a frequency error $\Delta \omega$ and a phase offset $\delta$; that is, the receiver carrier is $\cos((\omega_c + \Delta \omega)t + \delta)$ or $\sin((\omega_c + \Delta \omega)t + \delta)$. 

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a. Show that the output of the upper receiver branch is
\[ m_1(t) \cos(\Delta \omega t + \delta) - m_2(t) \sin(\Delta \omega t + \delta), \]
instead of \( m_1(t) \), and the output of the lower receiver branch is
\[ m_1(t) \sin(\Delta \omega t + \delta) + m_2(t) \cos(\Delta \omega t + \delta), \]
instead of \( m_2(t) \).
b. Find the signal-to-noise-plus-interference power ratio (SINR) on each branch, where the noise-plus-interference power equals the power associated with the unwanted signal (signal \( m_2(t) \) in the upper branch and signal \( m_1(t) \) in the lower branch) added to the random noise power. Assume that the noise \( n(t) \) signal added to the QAM signal \( \phi_{\text{QAM}}(t) \) in Fig. 4.19) has flat PSD \( S_n(f) = N_0/2 \).

**Solution (10 points)**

a. When the carrier at the demodulator has frequency and phase offset,
\[
x_1(t) = 2(m_1(t) \cos \omega_t t + m_2(t) \sin \omega_t t) \cos((\omega_c + \Delta \omega)t + \delta) \\
= 2m_1(t) \cos \omega_t t \cos((\omega_c + \Delta \omega)t + \delta) + 2m_2(t) \sin \omega_t t \cos((\omega_c + \Delta \omega)t + \delta) \\
= m_1(t) \left( \cos((2\omega_c + \Delta \omega)t + \delta) + \cos(\Delta \omega t + \delta) \right) + \\
m_2(t) \left( \sin((2\omega_c + \Delta \omega)t + \delta) - \sin(\Delta \omega t + \delta) \right)
\]

Similarly,
\[
x_2(t) = 2(m_1(t) \cos \omega_t t + m_2(t) \sin \omega_t t) \sin((\omega_c + \Delta \omega)t + \delta) \\
= 2m_1(t) \cos \omega_t t \sin((\omega_c + \Delta \omega)t + \delta) + 2m_2(t) \sin \omega_t t \sin((\omega_c + \Delta \omega)t + \delta) \\
= m_1(t) \left( \sin((2\omega_c + \Delta \omega)t + \delta) + \sin(\Delta \omega t + \delta) \right) + \\
m_2(t) \left( \cos(\Delta \omega t + \delta) - \cos((2\omega_c + \Delta \omega)t + \delta) \right)
\]

After \( x_1(t) \) and \( x_2(t) \) are passed through a low-pass filter, the outputs are
\[
m'_1(t) = m_1(t) \cos(\Delta \omega t + \delta) - m_2(t) \sin(\Delta \omega t + \delta) \\
m'_2(t) = m_1(t) \sin(\Delta \omega t + \delta) + m_2(t) \cos(\Delta \omega t + \delta)
\]

b. The desired signal for the upper branch is \( m_1(t) \cos(\Delta \omega t + \delta) \). The noise after demodulation is \( n_1(t) = 2n(t) \cos((\omega_c + \Delta \omega)t + \delta) \), which is passed through a LPF to get \( n'_1(t) \) with PSD
\[
S_{n'_1}(f) = \begin{cases} 
N_0 & |f| \leq B \\
0 & |f| > B 
\end{cases}
\]

The interference power is the power of the unwanted signal \( m_2(t) \sin(\Delta \omega t + \delta) \). Therefore the signal-to-noise-plus-interference power ratio for the upper branch is
\[
\text{SINR}_1 = \frac{\langle m_1^2(t) \cos^2(\Delta \omega t + \delta) \rangle}{2N_0 B + \langle m_2^2(t) \sin^2(\Delta \omega t + \delta) \rangle},
\]
where \( \langle g(t) \rangle \) represents the time-averaged value \( \langle g(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t) \, dt \).

Similarly, for the lower branch,
\[
\text{SINR}_2 = \frac{\langle m_2^2(t) \cos^2(\Delta \omega t + \delta) \rangle}{2N_0 B + \langle m_1^2(t) \sin^2(\Delta \omega t + \delta) \rangle}.
\]