Sample Midterm Examination Problems

1. Filters.
   a. The signal \( x(t) = \cos^4 2\pi t \) is the input to an ideal low-pass filter with cutoff frequency \( f_0 = 3 \) Hz. Find the output \( y(t) \).
   b. Find the impulse response of the nonideal bandpass filter whose transfer function is shown below. The widths of the upper and lower bases of the trapezoids are \( B \) and \( 2B \), respectively.

   ![Diagram of filter transfer function]

2. Overmodulation. A signal \( m(t) = \sin 2\pi t \) is transmitted using AM modulation:
   \[ \varphi_{AM}(t) = (1 + km(t)) \cos 20\pi t. \]
   a. Does the bandwidth of \( \varphi_{AM}(t) \) depend on the modulation index \( k \)?
   b. Sketch the envelope of \( \varphi_{AM}(t) \) for \( k = 2 \).
   c. Can the signal \( m(t) \) be recovered when \( k > 1 \)? Explain briefly.

3. FM modulation. Consider the FM modulated signal
   \[ x(t) = \cos \left( 2\pi f_c t + 2\pi k_f \int_0^t m(u) \, du \right). \]
   The information signal is a square wave of period \( 2T_0 \):
   \[ m(t) = 5 \sum_{n=-\infty}^{\infty} (-1)^n \Pi((t - nT_0)/T_0). \]
   Let \( f_c = 10 \) kHz, \( k_f = 2\pi \cdot 200 \), and \( T_0 = 0.001 \).
   a. Use Carson’s rule to find the bandwidth of \( x(t) \). You will have to make a reasonable estimate of the bandwidth of \( m(t) \). Hint: the Fourier series for the square wave of period 1,
      \[ \sum_n (-1)^n \Pi(2(t - \frac{1}{2}n)) \)
      is
      \[ \frac{4}{\pi} \left( \cos 2\pi t - \frac{1}{3} \cos 2\pi 3t + \frac{1}{5} \cos 2\pi 5t - \frac{1}{7} \cos 2\pi 7t + \cdots \right) \]
   b. Sketch the instantaneous frequency \( f_i(t) \) of \( x(t) \).
   c. Sketch or describe \( x(t) \).
   d. Suppose that this signal is demodulated with a zero-crossing detector. Find the minimum time interval \( T \) of the zero-crossing detector such that there are at least 2 zero crossings in every interval. For this \( T \), will there ever be more than 2 zero crossings?

4. Multipath channels. In a multipath channel, signal reflections give rise to multiple copies of the signal arriving at the receiver at different times. Depending on the time difference between these different arrivals, the signal may or may not be recoverable.
Consider the time-limited signal \( x(t) \) defined by
\[
x(t) = \begin{cases} 
    \cos(2\pi t/T) & |t| < T \\
    0 & |t| > T
\end{cases}
\]
Suppose that the signal is passed through a channel with frequency response
\[
H(f) = 1 + \cos(2\pi T_0 f),
\]
resulting in channel output \( y(t) = h(t) \ast x(t) \).

a. Find an expression for \( X(f) \) and sketch \( X(f) \). The plot sketches should indicate the general shape of the function and all key values on the \( x- \) and \( y- \)axes.

b. Find \( h(t) \). How many received copies of the signal with different delays result from this channel?

c. Sketch \( y(t) \) for \( T_0 = 3T \). How would you recover \( x(t) \) from \( y(t) \) ?

d. What is the minimum value for \( T_0 \) such that \( x(t) \) can be recovered from \( y(t) \) with no distortion?

5. Demodulation by sampling. In the communication system shown in the following figure, the signal \( w(t) \) is obtained by sampling the channel output at intervals of \( T \) seconds with impulses of area 1.

The information signal has Fourier transform
\[
X(f) = \begin{cases} 
    \frac{2}{3} & |f| < B/2 \\
    1 & B/2 \leq |f| < B \\
    0 & \text{otherwise}
\end{cases}
\]
The frequency response of the channel is
\[
H_1(f) = \begin{cases} 
    3 & |f| - f_c < B/2 \\
    1 & B/2 \leq |f| - f_c < B \\
    0 & \text{otherwise}
\end{cases}
\]
The carrier frequency \( f_c \) is much larger than the bandwidth \( B \) of \( x(t) \).

a. Sketch \( Z(f) \).

b. Sketch \( V(f) \).

c. Find \( v(t) \).

d. The sampled signal \( w(t) \) and its Fourier transform are
\[
w(t) = \sum_{n=-\infty}^{\infty} v(nT) \delta(t - nT) \quad \text{and} \quad W(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} V \left( f - \frac{n}{T} \right)
\]
Sketch \( W(f) \) when \( T = 1/f_c \).

e. Find \( H_2(f) \) so that \( y(t) = x(t) \) when \( T = 1/f_c \).