Square wave with period $2\pi$.

Square wave with period $2\pi$ is determined by values in $[-\pi, \pi]$:

$$w(t) = \begin{cases} 1 & |t| < \pi/2 \\ 0 & -\pi < |t| < \pi/2 \end{cases}$$

Square wave is a 1-bit quantization of cosine:

$$w(t) = \begin{cases} 1 & \cos t > 0 \\ 0 & \cos t < 0 \end{cases}$$

Crystal oscillators for digital circuits convert sinusoidal outputs to square waves using comparator and amplifiers.
Fourier Series Examples (cont.)

If \( n \neq 0 \),

\[
D_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} w(t) \, dt = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1 \cdot e^{-int} \, dt
\]

\[
= \frac{1}{2\pi} \frac{e^{-in\pi/2}}{-in} \bigg|_{-\pi/2}^{\pi/2} = \frac{1}{2\pi} \frac{e^{\pi in/2} - e^{-\pi in/2}}{in}
\]

\[
= \frac{\sin \pi n/2}{\pi n/2} = \text{sinc}(n/2) = (-1)^{n-1} \frac{2}{\pi n}
\]

If \( n = 0 \),

\[
D_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1 \cdot e^{-i0t} \, dt = \frac{1}{2\pi} \cdot \pi = \frac{1}{2}
\]
Fourier Series Coefficients of Square Wave
Trigonometric Series Approximating Square Wave

- $N = 3, \ N = 9$

- $N = 11, \ N = 19$
The Fourier series representation is

\[ g(t) = 0.504 \left( 1 + \sum_{n=1}^{\infty} \frac{2}{2 + 16n^2} \left( \cos 2nt + 4n \sin 2nt \right) \right) \]

The phase is \( \theta_n = -\tan^{-1} 4n \). Note that \( \lim_{n \to \infty} \theta_n = -\pi/2 \).

This calculation was done offline by copying from textbook. Other approved approaches are WWW and Mathematica.
Types of Systems

For theoretical and practical reasons, we restrict attention to systems and have useful properties and represent the physical world.

- Causal
- Continuous
- Stable
- Linear
- Time invariant

Fundamental fact: every linear, time-invariant system (LTIS) is characterized by

- Impulse response: \( w(t) = h(t) \ast v(t) \)
- Transform function: in frequency domain, \( W(f) = H(f) \cdot V(f) \)
Signals as Vectors

A signal \( g(t) \) defined for a finite number of time variables,

\[
g(t_k) = g_k, \quad k = 1, \ldots, n
\]
can be considered to be a vector of dimension \( n \):

\[
g = (g_1, g_2, \ldots, g_n)
\]

The Euclidean \textit{norm} (or size or magnitude) is

\[
\|g\| = \sqrt{|g_1|^2 + \cdots + |g_n|^2} = \left( \sum_{k=1}^{n} |g_k|^2 \right)
\]

(This definition works for complex-valued signals.)

The norm is used to measure how far apart are two signals, e.g., to tell how good an estimate is:

\[
\text{square error} = \|\hat{g} - g\|^2 = \left( \sum_{k=1}^{n} (|\hat{g}_k - g_k|^2) \right)
\]
Orthogonality

If two signals are orthogonal, then by Pythagoras’ theorem,

$$\|f\|^2 + \|g\|^2 = \|f + g\|^2$$

$$= (f + g) \cdot (f + g)$$

$$= f \cdot f + f \cdot g + g \cdot f + g \cdot g = \|f\|^2 + \|g\|^2 + 2(f \cdot g)$$

Thus two signals are orthogonal if and only inner product $f \cdot g = 0$. (In this case the energy of the sum is the sum of the energies.)

Recall that $f \cdot g = \|f\|\|g\| \cos \theta$, where $\theta$ is the angle between $f$ and $g$.

- $\theta = 0 \Rightarrow f$ and $g$ point in exactly the same direction
- $0 < \theta < \pi/2 \Rightarrow f$ and $g$ point in the same general direction
- $\theta = -\pi/2 \Rightarrow f$ and $g$ are perpendicular
- $\pi/2 < \theta < \pi \Rightarrow f$ and $g$ point in opposite general direction
- $\theta = -\pi \Rightarrow f$ and $g$ point in exactly the opposite direction
Signals as Vectors, II

For most signal processing applications, the number of samples is much larger than 3, so we cannot easily visualize signals as vectors.

In fact, \( n \) might be infinite, e.g., \( t_k = k \Delta t \) for \( k = 0, 1, 2, \ldots \) or even \( -\infty < k < \infty \).

When the dimension is infinite, the signal’s norm may be infinite. If

\[
\|g\|^2 = \sum_k |g_k|^2 < \infty
\]

the signal has finite energy and is said to belong to \( L^2 \).

The energy of the sampled signal depends on the sampling interval \( \Delta t \). The power in each interval is \( \|g_k\|^2 \), so total energy is

\[
\sum_k |g_k|^2 \Delta t
\]

Gauss determined the orbit of Ceres using only 24 samples and an early version of the FFT.
If the sample interval goes to zero, we obtain a continuous-time signal with energy

\[ \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \lim_{\Delta t \to 0} \sum_k |g_k|^2 \Delta t \]

The class of finite energy signals is named \( L^2 \) or \( L^2(-\infty, +\infty) \).

The two major tasks of signal processing are estimation and detection.

- Estimation: finding a good estimate \( \hat{g}(t) \) of an unknown signal \( g(t) \) using information of other signals
- Detection: identifying which of a finite number of possible signals is present, again based on signals that depend on \( g(t) \).

In communication systems, the available signal is the output of a channel:

\[ w(t) = h(t) \ast v(t) + z(t), \]

where the channel attenuates and distorts the input and adds noise.
Component of a Vector along Another Vector

A simple example of a channel with noise results in output

\[ g = cx + e \]

Where the gain \( c \) and error \( e \) are unknown. There are infinitely many solutions. To minimize error, we choose

\[ e = g - cx \]

to be orthogonal to \( g \).
The component of \( g \) along \( x \) is \( \|g\| \cos \theta \). Therefore
\[
c \|x\|^2 = \|x\| \|g\| \cos \theta = \langle g, x \rangle
\]
So we can solve for \( c \):
\[
c = \frac{\langle g, x \rangle}{\|x\|^2} = \frac{\langle g, x \rangle}{\langle x, x \rangle}
\]
This answer applies to continuous-time signals. Suppose \( g(t) \) is defined for \( t_1 < t < t_2 \). We can approximate \( g(t) \) by \( cx(t) \). The energy of the error
\[
E_e = \int_{t_1}^{t_2} |g(t) - cx(t)|^2 dt
\]
is minimized by solving \( \partial E_e / \partial c = 0 \):
\[
c = \frac{\int_{t_1}^{t_2} g(t)x(t) dt}{\int_{t_1}^{t_2} x(t) dt} = \frac{1}{E_x} \int_{t_1}^{t_2} g(t)x(t) dt
\]
Consider a radar pulse $g(t)$.

The return signal depends on whether the transmit signal encounters a target. Here $w(t)$ represents noise and interference.

$$y(t) = \begin{cases} 
  a g(t - t_0) & \text{target present} \\
  w(t) & \text{target absent}
\end{cases}$$

Typical pulse is $1 \mu s$ at $3$ GHz; repetition rates from $2$ kHz to $200$ KHz.
Application to Signal Detection, II

A key assumption (usually justified) is that noise $w(t)$ is *uncorrelated* with signal.

$$\langle w(t), g(t - t_0) \rangle = \int_{t_1}^{t_2} w(t)g(t - t_0)\, dt = 0$$

To see if a target is present at distance corresponding to delay $t_0$, we *correlate* the return signal with a delayed version of the radar pulse:

$$\langle y(t), g(t - t_0) \rangle = \begin{cases} 
\int_{t_1}^{t_2} a|g(t - t_0)|^2\, dt & \text{target present} \\
0 & \text{target absent} 
\end{cases}$$

Obviously, we decide based on whether the correlation is nonzero.

To detect targets at all possible distances, we vary $t_0$. 
Parseval’s Theorem

Complex exponentials are orthogonal. Suppose \( m \) and \( n \) are integers and \( m \neq n \). Suppose period \( T_0 = 1 \). Then

\[
e^{j2\pi mt} \cdot e^{j2\pi nt} = \int_0^1 e^{j2\pi mt} e^{-j2\pi nt} \, dt = \int_0^1 e^{j2\pi (m-n)t} \, dt
\]

\[
= \frac{e^{j2\pi (m-n)t}}{j2\pi (m-n)} \bigg|_0^1 = \frac{1 - 1}{j2\pi (m-n)} = 0
\]

It follows that the power of a Fourier series is sum of powers of terms:

\[
\|g(t)\|^2 = \int_0^1 |g(t)|^2 \, dt
\]

\[
= \left\| \sum_{n=-\infty}^{\infty} D_n e^{j2\pi nt} \right\|^2 = \sum_{n=-\infty}^{\infty} \|D_n e^{j2\pi nt}\|^2 = \sum_{n=-\infty}^{\infty} \|D_n\|^2
\]

We can calculate power in either time or frequency domain.

Also called Parseval’s identity (1799), Plancherel’s theorem (1910) and Rayleigh’s theorem.
Negative Frequencies

We cannot tell by looking at sine or cosine individually which direction the signal is moving. However, complex exponentials involve by sine and cosine, so we can tell which way the signal is rotating. By convention, positive frequency corresponds to counterclockwise, negative frequency to clockwise.