Review: Fourier Series Notations

- **Trigonometric series:**
  \[
  a_0 + \sum_{n=1}^{\infty} \left( a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right)
  \]
  or
  \[
  a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi n f_0 t + \sum_{n=1}^{\infty} b_n \sin 2\pi n f_0 t
  \]

- **Compact trigonometric series:**
  \[
  C_0 + \sum_{n=1}^{\infty} C_n \sin(2\pi n f_0 t + \theta_n)
  \]

- **Exponential Fourier series:**
  \[
  \sum_{n=-\infty}^{\infty} D_n e^{j2\pi n f_0 t}
  \]
Review: Fourier Series of Square Wave

Square wave with period $2\pi$.

Complex exponential Fourier series coefficients:

$$D_n = \begin{cases} 
\frac{1}{2} & n = 0 \\
0 & n \text{ even} \\
(-1)^{\frac{n-1}{2}} \frac{2}{\pi n} & n \text{ odd}
\end{cases}$$
Parseval’s Theorem

Complex exponentials are orthogonal. Suppose $m$ and $n$ are integers and $m \neq n$. Suppose period $T_0 = 1$. Then

$$e^{j2\pi mt} \cdot e^{j2\pi nt} = \int_0^1 e^{j2\pi mt} e^{-j2\pi nt} \, dt = \int_0^1 e^{j2\pi (m-n)t} \, dt$$

$$= \frac{e^{j2\pi (m-n)t}}{j2\pi (m-n)} \bigg|_0^1 = \frac{1 - 1}{j2\pi (m-n)} = 0$$

It follows that the power of a Fourier series is sum of powers of terms:

$$\|g(t)\|^2 = \int_0^1 |g(t)|^2 \, dt$$

$$= \left\| \sum_{n=-\infty}^{\infty} D_n e^{j2\pi nt} \right\|^2 = \sum_{n=-\infty}^{\infty} \|D_n e^{j2\pi nt}\|^2 = \sum_{n=-\infty}^{\infty} \|D_n\|^2$$

We can calculate power in either time or frequency domain.

Also called Parseval’s identity (1799), Plancherel’s theorem (1910) and Rayleigh’s theorem

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Negative Frequencies

We cannot tell by looking at sine or cosine individually which direction the signal is moving. However, complex exponentials involve by sine and cosine, so we can tell which way the signal is rotating. By convention, positive frequency corresponds to counterclockwise, negative frequency to clockwise.
Fourier Transform Motivated

Periodic functions have Fourier series. An aperiodic function is limit of periodic functions as $T \to \infty$.

As $T \to \infty$, $f_0 \to 0$, but coefficients have a common envelope.
Fourier Transform Motivated (cont.)

For finite $T_0$, Fourier series coefficients are

$$C_n = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} g(t) e^{-j2\pi f_0 nt} dt$$

As $T_0$ increases, the harmonic frequencies $nf_0$ become closer together. In the limit, the frequencies are continuous.

If $g(t)$ has finite energy, then the integral

$$\lim_{T_0 \to \infty} \int_{-T_0/2}^{+T_0/2} g(t) e^{-j2\pi f_0 nt} dt = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

exists and is finite for “most” values of $f = 2\pi \omega$.

We drop the normalizing factor $1/T_0$ so that the limit is nonzero.
Fourier Transform Definition

If $g(t)$ is a real- or complex-valued signal, its Fourier transform is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} \, dt.$$  

The signal is a continuous sum of complex exponentials:

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} \, df.$$  

The signal is approximated by sum of complex exponentials with

$$C_n = G(n\Delta f).$$
Fourier Transform Existence

Fact: if $g(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |g(t)| \, dt < \infty$$

then $G(f)$ exists for every frequency $f$ and is continuous.

Fact: if $g(t)$ has finite energy, i.e.,

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$

then $G(f)$ exists for “most” frequencies $f$ and has finite energy.

Fact: if $g(t)$ is periodic and has a Fourier series, then

$$G(f) = \sum_{n=-\infty}^{\infty} G(nf_0) \delta(f - nf_0)$$

is a weighted sum of impulses in frequency domain.
Fourier Transform Examples

One-sided exponential decay is defined by \( e^{-at}u(t) \) with \( a > 0 \):

\[
g(t) = \begin{cases} 
0 & t < 0 \\
e^{-at} & t > 0 
\end{cases}
\]

The Fourier transform of one-sided decay is (simple calculus):

\[
G(f) = \int_0^\infty e^{-2\pi j ft} e^{-at} \, dt = \int_0^\infty e^{-2\pi j ft - at} \, dt
\]

\[
= \int_0^\infty e^{(-2\pi j f-a)t} \, dt = \left[ \frac{e^{(-2\pi j f-a)t}}{-2\pi j f - a} \right]_{t=0}^{t=\infty}
\]

\[
= \frac{e^{-2\pi j f t}}{-2\pi j f - a} e^{-at} \bigg|_{t=\infty} - \frac{e^{-2\pi j f-a t}}{-2\pi j f - a} \bigg|_{t=0} = \frac{1}{2\pi j f + a}
\]

Since \( g(t) \) has finite area, its transform is continuous.

\( g(t) \) is real but its transform is complex valued. This is the usual situation.
Fourier Transform Examples (cont.)

We can rationalize \( G(f) \).

\[
G(f) = \frac{1}{a + 2\pi j f} = \frac{a - 2\pi i f}{a^2 + 4\pi^2 f^2} = \frac{a}{a^2 + 4\pi^2 f^2} - \frac{2\pi f}{a^2 + 4\pi^2 f^2}
\]

We can better picture \( G(f) \) using polar representation.

\[
|G(f)| = \frac{1}{\sqrt{a^2 + 4\pi^2 f^2}}, \quad \theta_G(f) = \angle G = -\tan^{-1}\left(\frac{-2\pi f}{a}\right)
\]
Fourier Transform Examples (cont.)

Fourier transform at $f = 0$:

$$G(0) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi \cdot 0 \cdot t} \, dt = \int_{-\infty}^{\infty} g(t) \, dt$$

is the area under $g(t)$, called the DC value.

For $g(t) = e^{-at}u(t)$, $G(0) = \frac{1}{a}$ is the only real value and largest in magnitude.

For one-sided decay, $G(-f) = G^*(f)$, complex conjugate of $G(f)$. This is true for all real-valued signals.

$$G^*(f) = \left( \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} \, dt \right)^* = \int_{-\infty}^{\infty} g(t)^* (e^{-j2\pi ft})^* \, dt$$

$$= \int_{-\infty}^{\infty} g(t) e^{j2\pi ft} \, dt = \int_{-\infty}^{\infty} g(t) e^{-j2\pi (-f)t} \, dt = G(-f)$$

We need only positive frequencies for real-valued signals.
Fourier Inversion Theorem and Duality

Fourier inversion theorem: if \( g(t) \) has a Fourier transform \( G(f) \),

\[
g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} \, df
\]

The inverse Fourier transform is defined by

\[
x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df
\]

The inverse transform differs from the forward transform in the sign of the exponent.

If \( \mathcal{F} \) is the system that produces the Fourier transform, then

\[
\mathcal{F}\{g(t)\} = G(f) \Rightarrow \mathcal{F}\{G(t)\} = g(-f)
\]

The forward transform results in the reversal of the inverse transform. This is called the principle of duality.
Important Fourier Transforms

The unit rectangle function $\Pi(t)$ is defined by

$$g(t) = \begin{cases} 
1 & |t| < \frac{1}{2} \\
0 & |t| > \frac{1}{2}
\end{cases}$$

Its Fourier transform is

$$G(s) = \int_{-\infty}^{\infty} e^{-2\pi j ft} \Pi(t) \, dt = \int_{-1/2}^{1/2} e^{-2\pi j ft} \cdot 1 \, dt = \frac{\sin \pi f}{\pi f}.$$  

Fact: every finite width pulse has a transform with unbounded frequencies.
Sinc

The sinc function is very important. Sadly, it has two definitions.

\[ \text{sinc } t = \frac{\sin t}{t} \quad \text{and} \quad \text{sinc } t = \frac{\sin \pi t}{\pi t} \]

The two definitions are stretched versions of each other. (Mathematica uses the former, \texttt{MATLAB} uses the latter.)

Basic properties of sinc (using first definition)

- \( \text{sinc } 0 = \lim_{t \to 0} \frac{\sin t}{t} = 1 \).
- \( \text{sinc } 2\pi n = 0 \) if \( n \neq 0 \).
- \( \int_{-\infty}^{\infty} |\text{sinc } t| \, dt = \infty \)
- \( \int_{-\infty}^{\infty} \text{sinc}^2 t \, dt = 1 \); thus \( \text{sinc } t \) has a finite energy Fourier transform
- By duality, Fourier transform of \( \text{sinc } t \) is \( \Pi(f) \).

\( \text{sinc } t \) is a band-limited pulse with no frequency content for \( f > \frac{1}{2} \).
Fourier Transform Time Scaling

If $a > 0$ and $g(t)$ is a signal with Fourier transform $G(f)$, then

$$\mathcal{F}g(at) = \int_{-\infty}^{\infty} g(at)e^{-j2\pi ft} = \int_{-\infty}^{\infty} g(u)e^{-j2\pi fu/a} (du/a) = \frac{1}{a} G\left(\frac{f}{a}\right)$$

If $a < 0$, change of variables requires reversing limits of integration:

$$\mathcal{F}g(at) = -\frac{1}{a} G\left(\frac{f}{a}\right)$$

Combining both cases:

$$\mathcal{F}g(at) = \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

Special case: $a = -1$. The Fourier transform of $g(-t)$ is $G(-f)$.

Compressing in time corresponds to expansion in frequency (and reduction in amplitude) and vice versa.

The sharper the pulse the wider the spectrum.
The Fourier transform of a squeezed rectangle function $\Pi(t/\tau)$ is

$$\tau \text{sinc}(\tau f).$$

Thus the transform of a narrow rectangular pulse of area 1 is

$$\mathcal{F}\left\{\frac{1}{\tau} \Pi(t/\tau)\right\} = \text{sinc}(\tau f)$$

In the limit, the pulse is the unit impulse, and its transform is the constant 1.

We can find the Fourier transform directly:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft} dt = e^{-j2\pi ft}\bigg|_{t=0} = e^{-j2\pi f\cdot0} = 1$$

The impulse is the mathematical abstraction of signal whose Fourier transform has magnitude 1 and phase 0 for all frequencies.

By duality, $\mathcal{F}\{1\} = \delta(f)$. All DC, no oscillation.
Important Fourier Transforms (cont.)

- Shifted impulse $\delta(t - t_0)$:
  \[
  \mathcal{F}\{\delta(t - t_0)\} = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi ft} \, dt = e^{-j2\pi ft_0}
  \]
  This is a complex exponential in frequency. By duality,
  \[
  \mathcal{F}\{e^{j2\pi f_0 t}\} = \delta(f - f_0)
  \]

- Sinuoids:
  \[
  \mathcal{F}\{\cos 2\pi f_0 t\} = \mathcal{F}\{\frac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\} = \frac{1}{2} \delta(f - f_0) + \frac{1}{2} \delta(f + f_0)
  \]
  \[
  \mathcal{F}\{\sin 2\pi f_0 t\} = \mathcal{F}\{\frac{1}{2i}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\} = \frac{1}{2i} \delta(f - f_0) - \frac{1}{2i} \delta(f + f_0)
  \]
  The impulse pairs at $+$ and $-$ frequencies correspond to two phases.

- Triangle function $\Delta(t)$:
  \[
  \Delta(t) = \begin{cases} 
  1 - 2|x| & |x| \leq \frac{1}{2} \\
  0 & \text{otherwise}
  \end{cases} \Rightarrow \mathcal{F}\{\Delta(t)\} = \frac{\sin(\pi f/2)^2}{(\pi f/2)^2}
  \]
  Triangle is smoother than rectangle, so its transform decreases faster.
Important Fourier Transforms (cont.)

- Laplacian pulse \( g(t) = e^{-a|t|} \) where \( a > 0 \). Since
\[
g(t) = e^{-at}u(t) + e^{at}u(-t),
\]
we can use reversal and additivity:
\[
G(f) = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + 4\pi^2 f^2}.
\]

- The signum function.
\[
\mathcal{F}\{\text{sgn}t\} = \frac{2}{j2\pi f} = \frac{1}{j\pi f}
\]

- The unit step function.
\[
\mathcal{F}\{u(t)\} = \frac{1}{2}\delta(t) + \frac{1}{j2\pi f}
\]

Note that \( u(t) - u(-t) = \text{sgn}t \).