Power Spectral Density of Line Codes (review)

- In general, the PSD of a line code is
  \[ S_y(f) = |P(f)|^2 S_x(f) \]
  where
  - \( S_x(f) \) is the power spectral density of the digital sequence \( \{a_k\} \)
    (The PSD of a digital sequence is periodic in frequency.)
  - \( P(f) \) is the PSD of the basic pulse

- Polar signaling.
  \[ S_y(f) = \frac{|P(f)|^2}{T_b} R_0 = \frac{|P(f)|^2}{T_b} \]

- AMI (bipolar) signaling for full-width pulses.
  \[ S_y(f) \frac{|P(f)|^2}{2T_b} (1 - \cos 2\pi T_b f) = \frac{|P(f)|^2}{T_b} \sin^2(\pi T_b f) \]
  
  For half-width pulses: AMI (bipolar) signaling for full-width pulses.
  \[ S_y(f) = \frac{|P(f)|^2 T_b}{4} \text{sinc}^2 \left( \frac{\pi f T_n b}{2} \right) \sin^2(\pi T_b f) \]
PSD of Polar Signaling (Matlab Experiment)
Split-Phase (Manchester) Coding

Manchester encoding is polar coding using a dipole pulse.

Bit value is direction of transition in middle of bit cell.

Advantages:
- DC value is 0
- Timing is easily obtained
PSD of Polar, Split-Phase, Bipolar RZ Signals

For a fair comparison, half-width pulses are used.
Intersymbol Interference (ISI)

Pulses transmitted over a physical channel (linear time-invariant system) are distorted—smoothed and stretched.

E.g., the impulse response of an RC-circuit is $h(t) = e^{-t/RC} u(t)$.

For $RC = 0.5$ the pulse response is

$$h(t) * \Pi(t - \frac{1}{2}) = \begin{cases} 0 & t < 0 \\ 1 - e^{-2t} & 0 < t < 1 \\ (1 - e^{-2})e^{-2t} & t > 1 \end{cases}$$
ISI Example: $h(t) = e^{-t/RC}/RC$
Reducing ISI: Pre-Emphasis

For different RC values:

- **RC = 0.5**
  - Input signal with ISI
  - Pre-emphasized signal

- **RC = 1.0**
  - Input signal with ISI
  - Pre-emphasized signal

- **RC = 2.0**
  - Input signal with ISI
  - Pre-emphasized signal
Reducing ISI: Transfer Functions and Equalization Filters
Equalization Impulse Responses, Signals after Equalization
Reducing ISI: Pulse Shaping

- A time-limited pulse cannot be bandlimited
- Linear channel distortion results in spread out, overlapping pulses
- Nyquist introduced three criteria for dealing with ISI.

The first criterion was that each pulse is zero at the sampling time of other pulses.

\[ p(t) = \begin{cases} 1 & t = 0 \\ 0 & t = \pm kT_b, \ k = \pm 1, \pm 2, \ldots \end{cases} \]

Harry Nyquist, “Certain topics in telegraph transmission theory”, Trans. AIEE, Apr. 1928
Pulse Shaping: \textit{sinc} Pulse

- Let $R_b = 1/T_b$. The \textit{sinc} pulse $\text{sinc}(\pi R_b t)$ satisfies Nyquist’s first criterion for zero ISI:

\[
\text{sinc}(\pi R_b t) = \begin{cases} 
1 & t = 0 \\
0 & t = \pm kT_b, \; k = \pm 1, \pm 2, \ldots 
\end{cases}
\]

- This pulse is bandlimited. Its Fourier transform is

\[
P(f) = \frac{1}{R_b} \Pi \left( \frac{f}{R_b} \right)
\]

- Unfortunately, this pulse has infinite width and decays slowly.
Nyquist Pulse

Nyquist increased the width of the spectrum in order to make the pulse fall off more rapidly.

The Nyquist pulse has spectrum width $\frac{1}{2}(1 + r)R_b$, where $0 < r < 1$.

If we sample the pulse $p(t)$ at rate $R_b = 1/T_b$, then

$$\bar{p}(t) = p(t) \Pi_{T_b}(t) = p(t)\delta(t) = \delta(t).$$

The Fourier transform of the sampled signal is

$$\bar{P}(f) = 1 = \sum_{k=-\infty}^{\infty} P(f - kR_b)$$
Nyquist Pulse (cont.)

Since we are sampling below the Nyquist rate $2R_b$, the shifted transforms overlap.

Nyquist’s criterion requires pulses whose overlaps add to 1 for all $f$.

![Diagram showing frequency spectrum]

For parameter $r$ with $0 < r < 1$, the resulting pulse has bandwidth

$$B_r = \frac{1}{2}(R_b + rR_b)$$

The parameter $r$ is called *roll-off factor* and controls how sharply the pulse spectrum declines above $\frac{1}{2}R_b$. 
Nyquist Pulse (cont.)

There are many pulse spectra satisfying this condition. e.g., trapezoid:

\[
P(f) = \begin{cases} 
1 & |f| < \frac{1}{2}(1 - r)R_b \\
1 - \frac{|f|-(1-r)R_b}{2R_b} & \frac{1}{2}(1 - r)R_b < |f| < \frac{1}{2}(1 + r)R_b \\
0 & |f| > \frac{1}{2}(1 - r)R_b 
\end{cases}
\]

A trapezoid is the difference of two triangles. Thus the pulse with trapezoidal Fourier transform is the difference of two \( \text{sinc}^2 \) pulses.

Example: for \( r = \frac{1}{2} \),

\[
P(f) = \frac{3}{2} \Lambda\left(\frac{f}{\frac{3}{2}R_b}\right) - \frac{1}{2} \Lambda\left(\frac{f}{\frac{1}{2}R_b}\right)
\]

so the pulse is

\[
p(t) = \frac{9}{4} \text{sinc}^2\left(\frac{3}{2}R_b t\right) - \frac{1}{4} \text{sinc}^2\left(\frac{1}{2}R_b t\right)
\]

This pulse falls off as \( 1/t^2 \).
Nyquist Pulse (cont.)

Nyquist chose a pulse with a “vestigial” raised cosine transform. This transform is smoother than a trapezoid, so the pulse decays more rapidly.

The Nyquist pulse is parametrized by $r$. Let $f_x = rR_b/2$. 
Nyquist Pulse (cont.)

Nyquist pulse spectrum is raised cosine pulse with flat porch.

\[
P(f) = \begin{cases} 
1 & |f| < \frac{1}{2}R_b - f_x \\
\frac{1}{2} \left( 1 - \sin \pi \left( \frac{f - \frac{1}{2}R_b}{2f_x} \right) \right) & |f| - \frac{1}{2}R_b < f_x \\
0 & |f| > \frac{1}{2}R_b + f_x 
\end{cases}
\]

The transform \( P(f) \) is differentiable, so the pulse decays as \( 1/t^2 \).
Nyquist Pulse (cont.)

Special case of Nyquist pulse is $r = 1$: full-cosine roll-off.

$$P(f) = \frac{1}{2}(1 + \cos \pi T_b f) \Pi(f / R_b)$$
$$= \cos^2 \left( \frac{1}{2} \pi T_b f \right) \Pi \left( \frac{1}{2} T_b f \right)$$

This transform $P(f)$ has a second derivative so the pulse decays as $1/t^3$.

$$p(t) = R_b \frac{\cos \pi R_b t}{1 - 4R_b^2 t^2} \text{sinc}(\pi R_b t) = \frac{\sin(2\pi R_b t)}{2\pi t(1 - 4R_b^2 t^2)}$$