Calculation of Driving Point Z

Fraction - λ Dipoles with Symmetrical $I(z)$

$\bar{J}(\bar{r})\, e^{+j\omega t}$ will be current density distribution on an antenna with an idealized source generator.

Some preliminaries:

$$\bar{H} = \frac{\nabla \times \bar{A}}{\mu}, \qquad \nabla \times \bar{H} = \nabla \times \nabla \times \frac{\bar{A}}{\mu}$$

$$\nabla \times \bar{H} = \nabla\left(\nabla \cdot \frac{\bar{A}}{\mu}\right) - \nabla^2 \frac{\bar{A}}{\mu} = \bar{J} + j\omega\epsilon\bar{E}$$

but $\nabla^2 \bar{A} + k^2 \bar{A} = -\bar{J}\mu \Rightarrow \nabla^2 \bar{A} = -\bar{J}\mu - k^2 \bar{A}$

eliminate $\bar{J}$ from above to obtain

$$\nabla(\nabla \cdot \bar{A}) + k^2 \bar{A} = j\omega\epsilon\mu\bar{E}$$

which relates $\bar{A}$, $\bar{E}$, and is useful in problems where $\bar{E}$ is the specified B.C.

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But still

$$\bar{A} = \frac{\mu}{4\pi} \int_{vol} \frac{\bar{J}(\bar{r}') e^{-jkR}}{R} dv', \quad \bar{R} = \bar{r} - \bar{r}',$$

as before

Combining the radiation integral and the last equation, above, we obtain <u>Pocklington's Equation</u>

$$(\nabla_{\bar{r}} \nabla_{\bar{r}} \cdot + k^2) \int_{vol} \mu \frac{\bar{J}(\bar{r}')}{4\pi R} e^{-jkR} dv' = j\omega\epsilon \frac{\mu}{\epsilon} \bar{E}$$

{ Diff Eq. in $\bar{A}$, Integral Eq in $\bar{J}$

$\nabla(\nabla \cdot \bar{A})$ differentiated w.r.t $\bar{r}$, not $\bar{r}'$

will represent impressed field due to sources.

With regard to the differentiation implied by

$$\text{"} \nabla_{\bar{F}} \nabla_{\bar{F}} \cdot \text{"}$$

Think of $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$,

<u>i.e.,</u> $R = R(x, x'; y, y'; z, z')$

Primed quantities represent source points, unprimed quantities represent observation points. The "unprimed $\bar{F}$" associated with $\nabla_{\bar{F}}$ means <u>differentiate</u> w.r.t. the <u>observation point</u>.

Strength and Utility of Pocklington's Eq. for Antennas

1. If $\bar{E}(\bar{r})$ is known at all points occupied by an antenna, then P. Eq. is an integral equation for the unknown current distribution $\bar{J}(\bar{r})$*.

Most antennas are composed of good conductors, so usually $\bar{E}(\bar{r})_{tang} = 0$, except in the vicinity of the source or driving transmission line, called the feed point.

* or knowing $\bar{J}(\bar{r})$ one can get $\bar{E}$, see below.

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2. If a good estimate of $\bar{E}(\bar{r})$ is available, then numerical techniques will provide the solution for the current distribution.

3. Knowing $\bar{J}(\bar{r}')$ one can obtain the driving point Z and the radiation pattern.

N.B. Previously, $\bar{r}$ and $\bar{r}'$ have refered to different regions of space. Here they will typically refer to or explore the same region of space.

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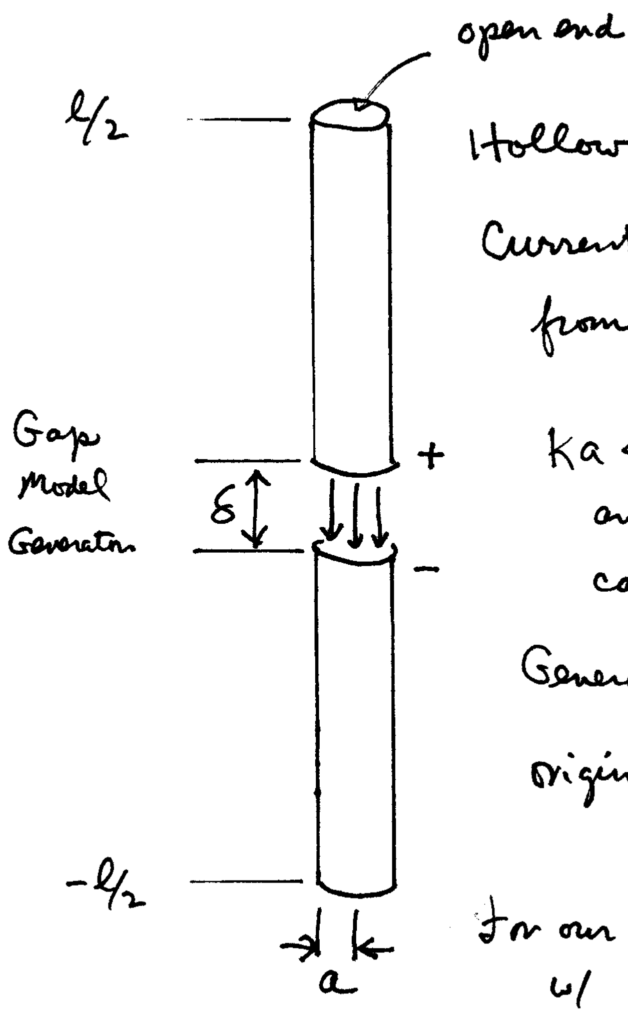
How to apply this to Dipole?



Hollow tube – current flows on outer walls.

Current is lineal, z-directed, on surface from $-\ell/2 \leftrightarrow -\frac{\delta}{2}$ and $\frac{\delta}{2} \leftrightarrow \ell/2$

$ka \ll 1$ so fields inside the tube and deviations from circular symmetry can be neglected.

Generator applies cylindrical $\bar{E}$ at the origin, $E_z = 0$ elsewhere.

For our purposes assume circular cross sections w/ radius a.

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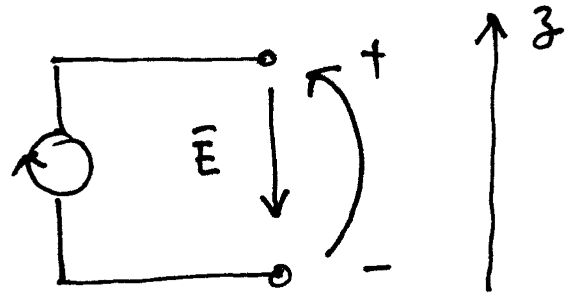
V is not a function of a, ϕ.

Voltage is positive



$\bar{E}$ is directed downward, towards negative z.

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As there are only z-directed fields present, presumably driving z-directed currents, P. Eq., above, becomes . . .

$\bar{A}$, for circular symmetry

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) \underbrace{\int \frac{K_z(\phi', z')}{4\pi R} e^{-jkR} a \, d\phi' dz'}_{\text{Surface}} = j\omega\epsilon_0 \mu_0 \underbrace{E_z(\phi, z)}_{\text{hollow generator}}$$

$E_z = 0$, except at the terminals. $V = -\int_{-\delta/2}^{\delta/2} E_z(a, z) \, dz = 1$

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With δ vanishing, $E_z(\phi, z)$ takes on the characteristics of $\delta(z)$ - a Dirac Delta.

$$\lim_{\delta \to 0} \int_{-\delta/2}^{\delta/2} E_z(z)\, dz = -1 \quad \text{(volts)} \Rightarrow \quad E_z(z) = 0, \; z \neq 0.$$

P. Eq. above is a general relationship between $\bar{J}(\bar{r})$ and $\bar{E}$. But how to solve it? We will apply diff. eq. for $A_z(z)$ to the dipole, since we know E_z there, under the assumption that $\delta \to 0$.

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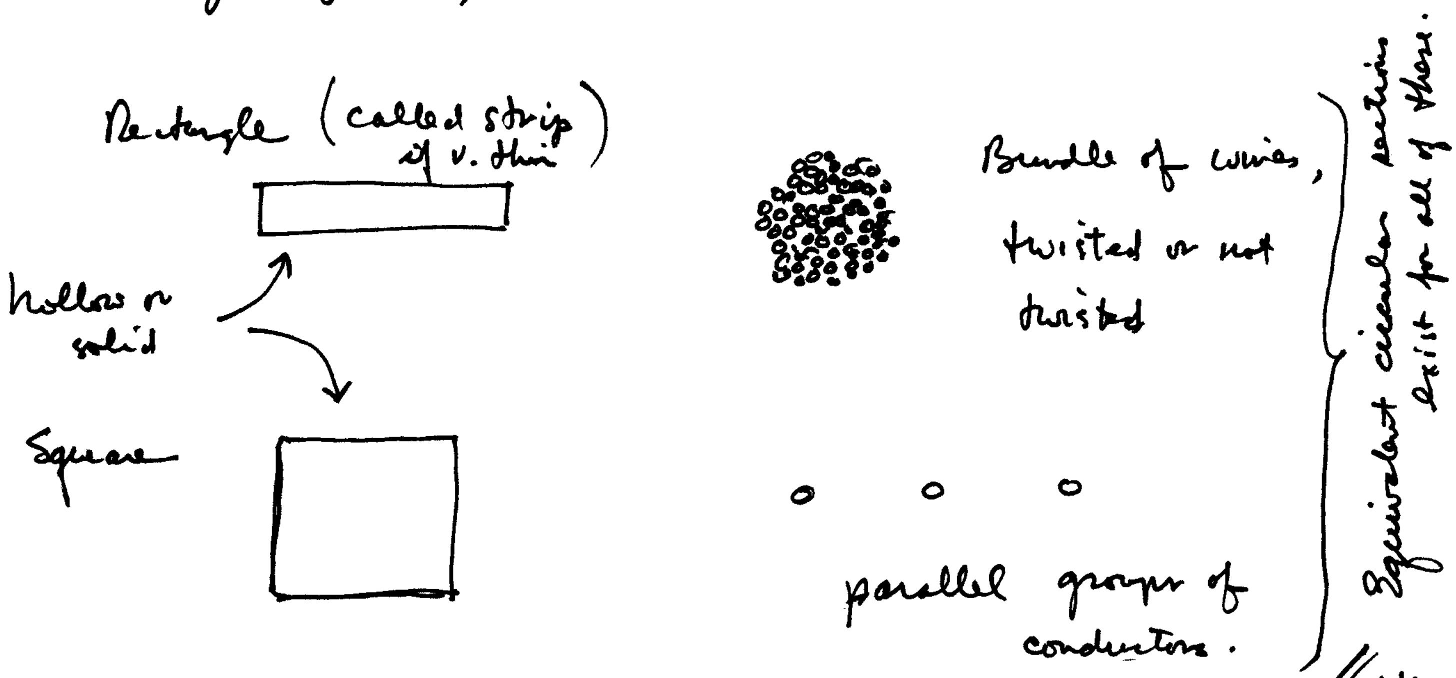
Here we have pictured the element as a right-circular cylinder. But this is not an important feature. Analysis can easily be carried out for any cylinder; other common cross sections might be



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(An Aside ...)

Clarification of Pocklington's Equation

Why does only $\frac{\partial^2}{\partial z^2}$ appear on LHS ?

This can be seen by plugging through ...

$$\nabla \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial}{\partial z}A_z$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{u}_\phi + \frac{\partial f}{\partial z}\hat{u}_z$$

$$\nabla(\nabla\cdot\bar{A}) = \nabla\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial}{\partial z}A_z\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial}{\partial z}A_z\right)\hat{u}_r$$

$$+ \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial}{\partial z}A_z\right)\hat{u}_\phi$$

$$+ \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial}{\partial z}A_z\right)\hat{u}_z$$

We are trying to solve the vector equation(s)

$$\nabla(\nabla\cdot\bar{A}) + k^2\bar{A} = j\omega\epsilon\bar{E}$$

or

$$\frac{\partial}{\partial r\partial z}A_z + k^2 A_r = j\omega\epsilon E_r \qquad (\hat{u}_r)$$

$$\frac{1}{r}\frac{\partial}{\partial\phi\,\partial z}A_z + k^2 A_\phi = j\omega\epsilon E_\phi \qquad (\hat{u}_\phi)$$

$$\boxed{\frac{\partial^2}{\partial z^2}A_z + k^2 A_z = j\omega\epsilon E_z \qquad (\hat{u}_z)}$$

$$\nabla(\nabla\cdot\bar{A}) = ?$$

$$\hat{A} = \hat{u}_z \int_0^{2\pi}\int_{-\ell/2}^{\ell/2} \frac{K_z(\phi',\bar{r}')}{4\pi R} e^{-jkR} a\, d\phi' dz'$$

$$\nabla(\nabla\cdot\bar{A}) = \frac{\partial}{\partial r\partial z} A_z\, \hat{u}_r + \frac{1}{r}\frac{\partial}{\partial\phi\partial z} A_z\, \hat{u}_\phi + \frac{\partial^2}{\partial z^2} A_z\, \hat{u}_z$$

But note that since we have z-directed currents only

$$A_r = A_\phi = 0$$

while Pocklington's Equation $\rightarrow$ A_z

given $\bar{A} = 0 \cdot \hat{u}_r + 0 \cdot \hat{u}_\phi + A_z \hat{u}_z$

$$E_r = \frac{1}{j\omega\epsilon} \frac{\partial}{\partial r \partial z} A_z$$

$$E_\phi = \frac{1}{j\omega\epsilon r} \frac{\partial}{\partial \phi \partial z} A_z$$

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Note that if only z-direction currents are present, then $\bar{A} = A_z \bar{a}_z$, and

$$j \omega \epsilon_0 E_z = \frac{\partial^2 A_z}{\partial z^2} + k^2 A_z \quad *$$

so specifying E_z is sufficient to determine $\bar{A}$, $\bar{J}$ (see above)

This does not imply that other components of $\bar{E}$ are necessarily $= 0$, only that they are not required to get $\bar{J}$.

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* Note that is a wave equation in A_z

(End aside)

That is, we will obtain the solution in two parts.

1. Solve the diff. eq. for $\bar{A}_z$.*

2. Solve the radiation integral for $\bar{J} \sim I(z)$.

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* Since the diff. eq. doesn't involve cross terms of vector components and because our boundary condition will be expressed in terms of E_z, we can solve for A_z independantly of A_x, A_y.

Solution:

$$A_z(z) = \xi_1 \cos kz + \xi_2 \sin kz \qquad z > 0$$
$$= \xi_3 \cos kz + \xi_4 \sin kz \qquad z < 0$$

is solution to the homogeneous wave equation

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) A_z(z) = 0,$$ which is OK except at $z = 0$.

at $z = 0$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) A_z(z) = j \omega \epsilon_0 \mu_0 E_z(z)$$

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For *finite* $A_z(z)$, singularity associated with $E_z(z)$ @ $z=0$ must be accommodated by

$$\frac{\partial^2}{\partial z^2} A_z(z) .$$

$$j\omega\epsilon_0\mu \int_{-\delta/2}^{\delta/2} E_z(z)\,dz = -j\omega\epsilon_0\mu \int_{-\delta/2}^{\delta/2} \frac{d^2 A_z}{dz^2}\,dz = \left.\frac{dA_z}{dz}\right|_{-\delta/2}^{\delta/2}$$

So the first derivative of A_z must change by $-j\omega\epsilon_0\mu$ across the source, for a unit strength source.

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$$\frac{d}{dz}\left(\xi_1 \cos kz + \xi_2 \sin kz\right) = -\xi_1 k \sin kz + \xi_2 k \cos kz \qquad z > 0$$

$$\frac{d}{dz}\left(\xi_3 \cos kz + \xi_4 \sin kz\right) = -\xi_3 k \sin kz + \xi_4 k \cos kz \qquad z < 0$$

$$z = \pm\frac{\delta}{2} \to 0, \quad k(\xi_2 - \xi_4) = -j\omega \overset{\mu_0}{\epsilon_0}; \quad \xi_2 - \xi_4 = -j\frac{\omega \overset{\mu}{\epsilon_0}}{k} = -j\overset{\mu_0}{/\eta_0}$$

By symmetry $\xi_2 = -\xi_4 = -j\overset{\mu_0}{/2\eta_0}$

and $\xi_1 = \xi_3 = C$.

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Why is this?

Symmetry forces $K_z(z)$ to be even, then

$$A_z(z) = \frac{1}{4\pi} \int_{\text{surface}} \frac{K_z(z')}{R} e^{-jkR} \, a \, d\phi' dz'$$

forces A_z to be symmetric as well.

Functional Form of A_z

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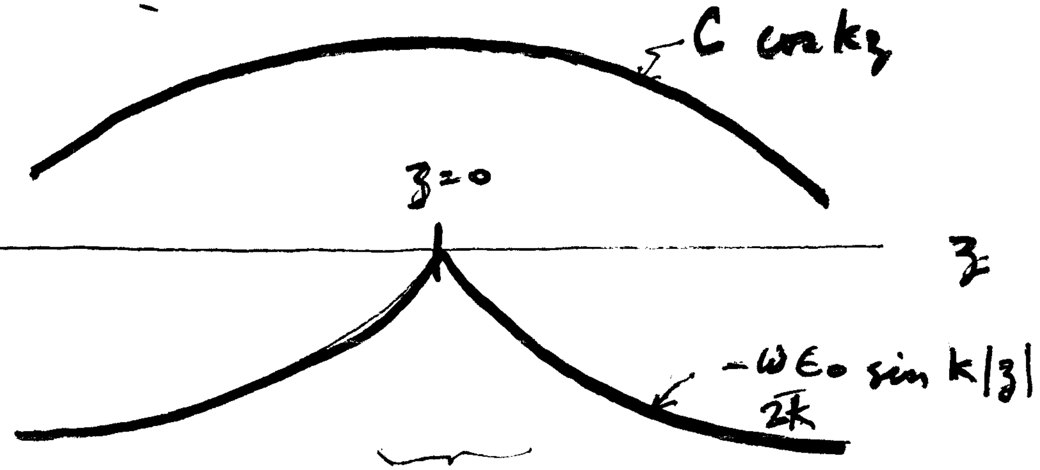
$$A_z(z) = C\cos kz - j\frac{\omega\epsilon_0}{2k}\sin k|z|$$



$$\int_{-\delta/2}^{\delta/2} \frac{d^2}{dz^2}A_z = -j\omega\epsilon_0 = \left.\frac{dA_z}{dz}\right|_{-\delta/2}^{\delta/2}$$

get C from

$$I(z)\Big|_{z=\pm \ell/2} = 0$$