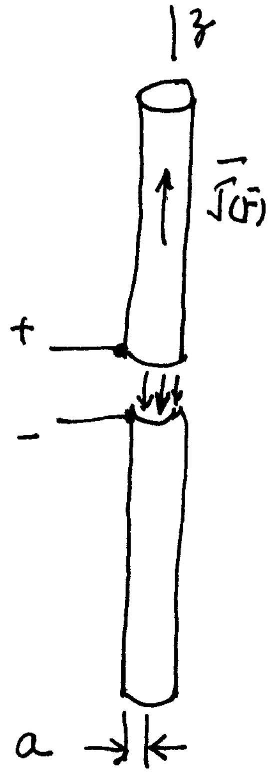
# How to Solve?




$$\underbrace{(\nabla_r \nabla_r \cdot + k^2) \int_{vol} \frac{\bar{J}(\bar{r}')\, e^{-jkR}}{4\pi R}\, dv' = j\omega\epsilon_0 \bar{E}}_{\text{Pocklington's Equation}}$$

$$\downarrow$$

$$\underbrace{\int_{-\ell/2}^{\ell/2} \frac{I(z')}{4\pi R}\, e^{-jkR}\, dz' = C \cos kz - j\,\frac{\omega\epsilon_0}{2k} \sin k|z| \qquad R^2 = a^2 + (z-z')^2 \;{}^{*}}_{\text{Hallen's Equation}}$$

* current on z-axis,
observation point on dipole, or v.v.

# Solutions to Hallen's Integral Equation

Rewrite equation in the form

$$\int_0^{\ell/2} G(z,z')\, I(z')\, dz' = C \cos kz - \frac{j V}{2\eta} \sin k|z|$$

where $G(z,z') = \dfrac{e^{-jkr}}{4\pi r} + \dfrac{e^{-jkr'}}{4\pi r'}$

$$r = \sqrt{a^2 + (z-z')^2}\,, \quad r' = \sqrt{a^2 + (z+z')^2}$$

## Moment Method Soln

$$\int_a^b G(z,z')\, f(z')\, dz' = g(z) \qquad \text{F.H. } 1^{\text{st}} \text{ kind}$$

Expand $f(z) \simeq c_1 f_1(z) + c_2 f_2(z) + \cdots c_N f_N(z) = \hat{f}(z)$

$f_n(z)$ are basis functions chosen heuristically —

$$\underbrace{\sum_{n=1}^{N} c_n \int G(z,z')\, f_n(z')\, dz'}\ \simeq g(z)$$

Forgiving of form of $\hat{f}_N(z)$ because of the integration. /...

/...

While the true soln $f(z)$ gives equality for every value of $z$, $\hat{f}(z)$ <u>cannot</u> do so. But it can do so <u>approximately</u>.

This approach is a <u>gross</u> simplification of classical orthogonal expansions, such as Fourier expansions.

It is also extremely efficacious!!
(and pretty accurate as well)

Point Matching Solution — the grossest form of MofM — enforces equality at fixed points, $z_m$,

$$\sum_{n=1}^{N} c_n \int_a^b G(z_m, z') f_n(z')\, dz' = g(z_m)$$

$$\sum a_{mn} c_n = b_m \quad , \quad c_n \text{ are unknowns}$$

$$a_{mn} = \int_a^b G(z_m, z') f_n(z')\, dz' \quad , \quad b_m = g(z_m)$$

in this case.

match point index

source point index

(more general) Testing function Approach

Also define a set of functions $\{W_i\} = W_1, W_2, \cdots W_N$

$W_i$ are called testing functions or weighting functions.

$$\sum_{n=1}^{N} c_n \underbrace{\int_a^b \int_a^b G(z,z') f_n(z') w_m(z) \, dz' \, dz}_{a_{mn}} = \underbrace{\int_a^b g(z) w_m(z) \, dz}_{b_m}$$

$$\sum_{n=1}^{N} c_n \, a_{mn} = b_m \quad \text{or} \quad [a_{mn}] c_n = b_m$$

/...

/ . . .

## Galerkin's Method

$$w_m = f_m$$

## Point-Matching Method

$$w_m(z) = \delta(z - z_m)$$

But other choices are possible.

## MoM

- based on mathematics of linear vector spaces and includes the theory of errors

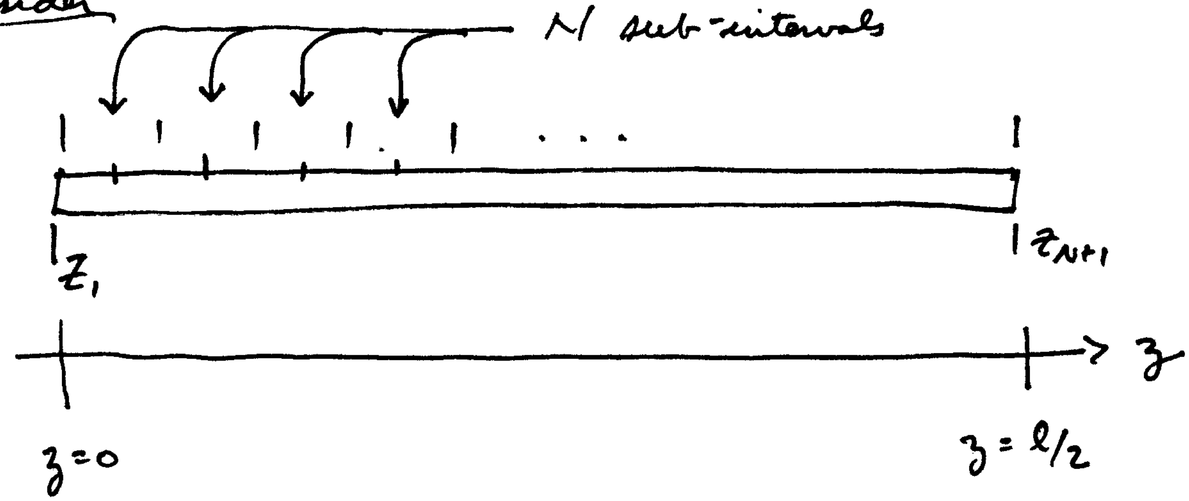
- still an active area of research

- generally validated by experiments

- being displaced by finite element methods
  - sparse matrices
  - efficient computation.

## Example: Application to Hallen's Eq. for dipole.

Consider




$$z_n = \frac{n-1}{N}\cdot\frac{\ell}{2}$$

$\Delta z_n = z_{n+1} - z_n$ is $n^{th}$ sub-interval

Use sub-sectional basis functions $\Rightarrow$ $p_n(z) = 1 \quad z \in \Delta z_n$

$\quad\quad = 0 \quad z \notin \Delta z_n$

these are just pulses !!!

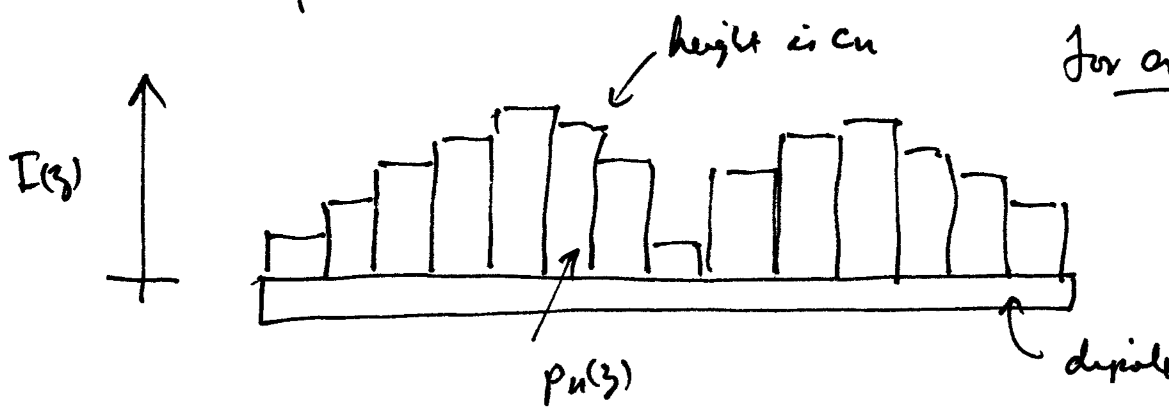
/...

/...

so $$I(z) \simeq \sum_{n=1}^{N} c_n \, p_n(z)$$ — not a complete set — but an orthogonal set!

We will attempt to solve for current on the dipole in terms of the stepped pulse approximation. This corresponds physically to an approximation by a series of ideal dipoles

For example



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How to form the equations

choose $z_m$ at midpoints of $\Delta z_n$.

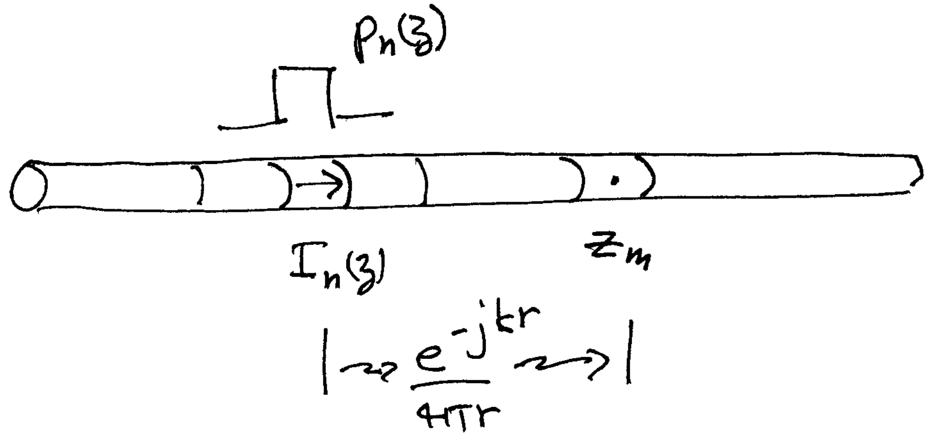
$$z_m = \left(\frac{2m-1}{2N}\right)\ell/2$$

divide length into 2N intervals and then take only the odd points, beginning with 1.

$$\sum_{n=1}^{N} c_n \int_0^{\ell/2} G(z_m, z')\, p_n(z')\, dz' = \underbrace{C \cos k z_m - \frac{j}{2\eta} \sin k|z_m|}_{A_z(z_m)}$$

$$a_{mn} = \int_{z_n'}^{z_{n+1}} G(z_m, z')\, dz'$$

/...




Sum of all such effects, $p_n(z)$ on $A_z(z_m)$

must equal total $A_z(z_m)$

/...

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ \vdots & & & & \\ \vdots & & & & \\ a_{N1} & & & & a_{NN} \end{bmatrix} \begin{bmatrix} c_1 \\ \\ \\ c_N \end{bmatrix} = \begin{bmatrix} C\cos k z_1 - \frac{j\mu}{2\eta}\sin k|z_1| \\ \vdots \\ \vdots \\ \cdot \end{bmatrix}$$

B.C. is $c_N = 0$

So this can be re-arranged and simplified further, as ...

/ . . .

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N-1} & d_1 \\ a_{21} & a_{22} & \cdots & a_{2N-1} & d_2 \\ \vdots & & & & \\ a_{N1} & a_{N2} & \cdots & a_{NN-1} & d_N \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ C \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$$

here is unknown C from expression for $A_z(z)$.

$$d_m = -\cos k z_m$$

$$b_m = -j^{\mu}/_{2\eta} \sin k z_M$$

find $C_1 \cdots C_{N-1}, C$ by standard methods.