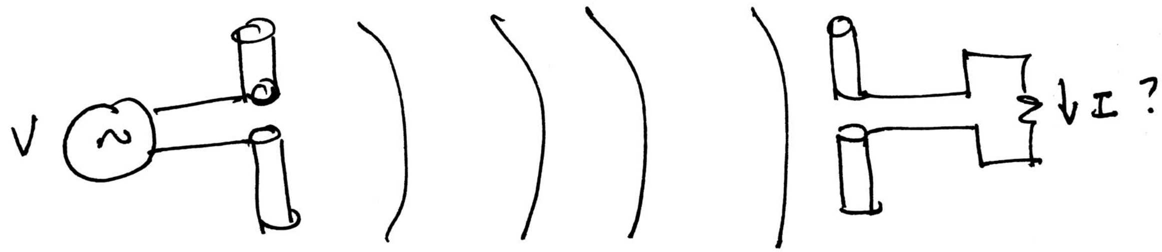
Maxwell's Equations

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu \frac{\partial \bar{H}}{\partial t} = -j\omega\mu \bar{H}$$

$$\nabla \times \bar{H} = +\frac{\partial \bar{D}}{\partial t} + \bar{J} = +j\omega\epsilon \frac{\partial \bar{E}}{\partial t} + \bar{J}$$

$$\nabla \cdot \bar{B} = 0 \quad , \quad \nabla \cdot \bar{D} = \rho$$



$$\bar{S} = \bar{E} \times \bar{H} \text{ watts}/m^2$$

Can this be simplified?

Express $\bar{E}, \bar{H}$ in terms of potentials.

Since $\nabla \cdot \bar{B} = 0$, $\bar{B} = \mu \bar{H} = \nabla \times \bar{A}$

$\mu \bar{H}, \bar{B}$ are solenoidal, i.e., they have no divergence

then $\nabla \times \bar{E} = -j\omega(\nabla \times \bar{A})$

$$\nabla \times (\bar{E} + j\omega \bar{A}) = 0$$

$\bar{E} + j\omega\bar{A}$ is "irrotational" (has no curl)

$\therefore \quad \bar{E} + j\omega\bar{A} = -\nabla\phi \; ; \quad \bar{E} = -j\omega\bar{A} - \nabla\phi$

$\bar{A}$ is magnetic vector potential,

ϕ is (scalar) electric potential.

/...

$\bar{A}$ has been specified ; $\nabla \cdot \bar{A}$ has not been specified

$$\bar{A} = \bar{A}_{sol} + \bar{A}_{irrot} \quad ; \quad \mu \bar{H} = \nabla \times \bar{A}$$

$$\nabla \times \bar{A} = \nabla \times \bar{A}_{sol} + \cancelto{0}{\nabla \times \bar{A}_{irrot}} = \nabla \times \bar{A}_{sol}$$

$$\nabla \cdot \bar{A} = \cancelto{0}{\nabla \cdot \bar{A}_{sol}} + \nabla \cdot \bar{A}_{irrot} = \nabla \cdot A_{irrot}$$

$\bar{A}_{sol}$ & $\bar{A}_{irrot}$ are independent as far as $\bar{H}$ is concerned ; $\bar{A}_{irrot}$ has no effect on $\bar{H}$

Choose $\nabla \cdot \bar{A} = -j\omega\epsilon\mu\Phi$

Lorentz Gauge

(Lorenz Gauge) /...

What is $\bar{A}$? How is it found?

$$\nabla \times \bar{H} = \nabla \times \left(\frac{\nabla \times \bar{A}}{\mu} \right) = \frac{1}{\mu} \nabla \times (\nabla \times \bar{A}) \; ; \quad \mu = \text{constant}$$

$$\frac{1}{\mu} \nabla \times \nabla \times \bar{A} = j\omega\epsilon \bar{E} + \bar{J} = j\omega\epsilon(-j\omega\bar{A} - \nabla\phi) + \bar{J}$$

vector ident: $\quad \nabla \times (\nabla \times \bar{A}) = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$

By substitution:

$$\nabla^2 \bar{A} + \omega^2 \mu\epsilon \bar{A} - \nabla(j\omega\epsilon\mu\phi + \nabla \cdot \bar{A}) = -\mu \bar{J}$$

non-homogeneous p.d. eq. for $\bar{A}$, ϕ?

$\mu \bar{J}$ is source function.

/...

What is ϕ?

$$-\nabla\phi = \bar{E} + j\omega\bar{A} \quad ; \quad \nabla\cdot\bar{A} = -j\omega\mu\epsilon\phi$$

$$-\nabla^2\phi = \nabla\cdot\bar{E} + j\omega\nabla\cdot\bar{A} = \nabla\cdot\bar{E} + \omega^2\mu\epsilon\phi$$

$$\nabla^2\phi + \omega^2\mu\epsilon\phi = -\nabla\cdot\bar{E} = -\nabla\cdot\frac{D}{\epsilon} = -\rho/\epsilon$$

$$\boxed{\nabla^2\phi + \omega^2\mu\epsilon\,\phi = -\rho/\epsilon}$$

Wave eqn in ϕ, driven by charge

scalar potential is driven by charge

vector " " " " current.

Effects of charge and current have been separated!

With $\nabla \cdot \bar{A} = -j\omega \epsilon \mu \Phi$

$$\nabla^2 \bar{A} + \omega^2 \mu \epsilon \bar{A} = -\mu \bar{J}$$

<u>Vector Helmholtz Eqn</u>

$$\nabla^2 \bar{A} = \nabla^2 A_x \hat{u}_x + \nabla^2 A_y \hat{u}_y + \nabla^2 A_z \hat{u}_z$$

$\therefore$ $\nabla^2 A_x + \omega^2 \mu \epsilon A_x = -\mu J_x$

$\nabla^2 A_y + \omega^2 \mu \epsilon A_y = -\mu J_y$

$\nabla^2 A_z + \omega^2 \mu \epsilon A_z = -\mu J_z$

Wave eqn. in each component of $\bar{A}$. Homogeneous p.d. eq if $\bar{J} = 0$

$\bar{A}$, Φ are decoupled by choice of $\nabla \cdot \bar{A}$

$\bar{A} \leftarrow \bar{J}$ alone.

/...

Fields

a) $\bar{E} = -j\omega\bar{A} - \nabla\phi \qquad \phi = -\dfrac{\nabla\cdot\bar{A}}{j\omega\mu\epsilon}$

$$\bar{E} = -j\omega\bar{A} + \frac{\nabla(\nabla\cdot\bar{A})}{j\omega\mu\epsilon} \; ; \; \bar{H} = \frac{1}{\mu}\nabla\times\bar{A}$$

$$= \frac{j}{\omega\mu}\nabla\times\bar{E}$$

b) alt.

$$\mu\bar{H} = \nabla\times\bar{A}$$

$$\bar{E} = \frac{1}{j\omega\epsilon}(\nabla\times\bar{H} - \bar{J}) \xrightarrow[J=0]{} \frac{1}{j\omega\epsilon}\nabla\times H$$

To find electric & magnetic fields $\bar{E}, \bar{H}$

1. Solve vector wave eqn. for $\bar{A}$, given $\bar{J}$

2. Apply one of several relationships between $\bar{A}, \bar{E}, \bar{H}, \bar{J}$

Step 1. <u>Sol'n to Vector Wave Equation</u> ?

Consider LaPlace's Equation

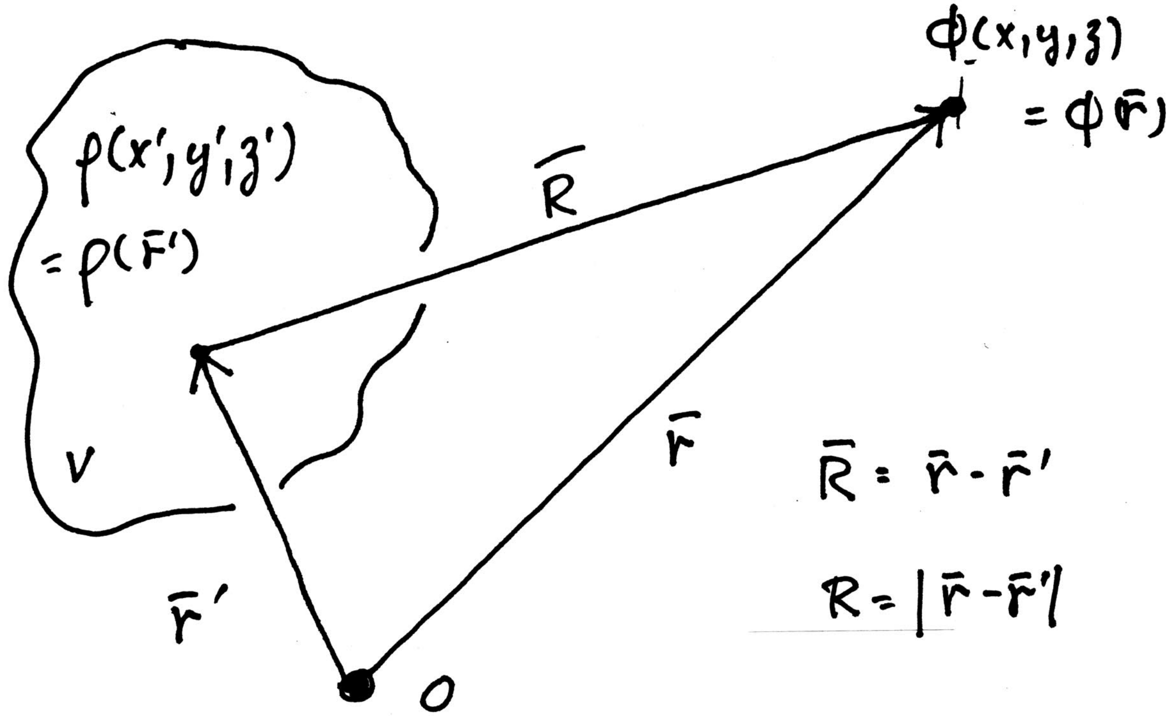
$$\nabla^2 \phi + 0 = -\rho/\epsilon \qquad \text{(statics)}$$

$$\phi = \int_{vol} \frac{\rho(x', y', z')}{4\pi \epsilon R} \, dv'$$

Basically a convolution integral of a point, or impulse response — a.k.a. a Green's function

/...

/... <u>Geometry</u>



$\bar{R} = \bar{r} - \bar{r}'$

$R = |\bar{r} - \bar{r}'|$

Convolution is $\rho(\bar{r})$ with $\dfrac{1}{4\pi\epsilon R}$, over V

Scalar Wave Equation

Sinusoidal steady state sol'n

$$\nabla^2 \phi + \omega^2 \mu \epsilon \phi = -\rho/\epsilon$$

$$\phi = \phi(x, y, z, t) = \phi(\bar{r}, t) = \int_{\text{volume}} \frac{\rho(x', y', z') e^{j(\omega t - kR)}}{4\pi\epsilon R} \, dv'$$

$$R = |\bar{r} - \bar{r}'|$$

Same as before except we use the "retarded" Green's function

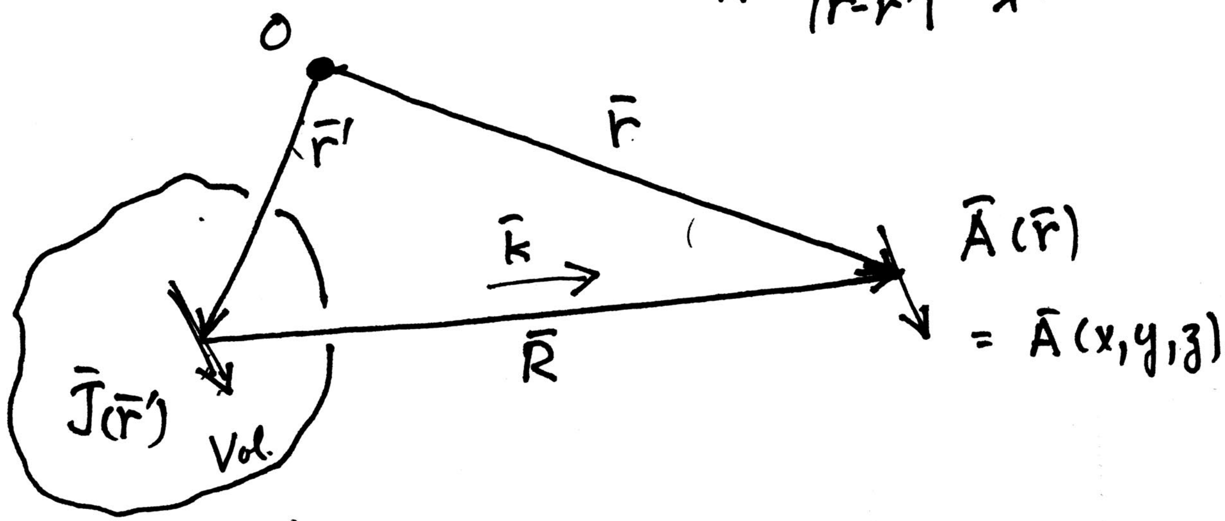
$$\frac{e^{j(\omega t - kR)}}{4\pi\epsilon R}, \quad k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \omega\sqrt{\mu\epsilon} \quad \begin{Bmatrix} \omega t - kR \\ \omega\left(t - \frac{R}{c}\right) \end{Bmatrix}$$

Similarly, for Vector Wave Equation

sinusoidal steady state

$$\bar{A}(x,y,z,t) = \int_{vol} \mu \frac{\bar{J}(x',y',z')}{4\pi R} e^{j(\omega t - kR)} \, dv'$$

$$\bar{k} = \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|} \cdot \frac{2\pi}{\lambda}$$



So —

Solution is a retarded vector superposition of the effects of all the differential elements of current $[J(\bar{r})]dv$, wherein every component of the vector current density $\bar{J}$ contributes to a like component of $\bar{A}$!!!

This provides us with a very direct means for visualizing the fields at 'great' distances from their source.

Implications for fields at great distances ?

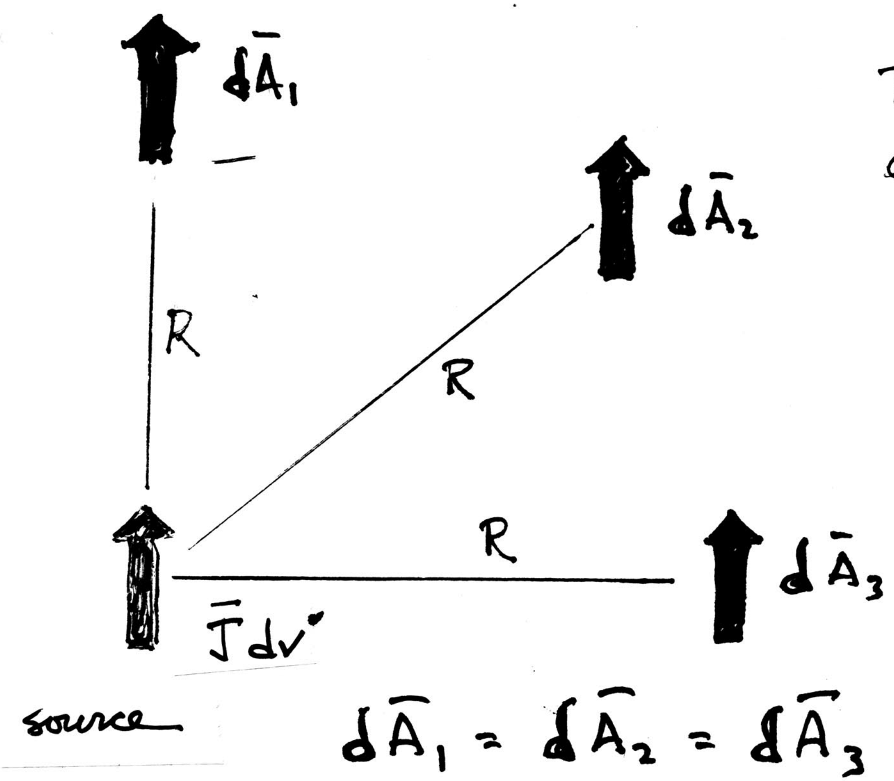
general
$$\begin{cases} \bar{H} = \dfrac{\nabla \times \bar{A}}{\mu}, \\[2ex] \bar{E} = -j\omega\cancel{\mu}\bar{A} - \nabla\phi = -j\omega\cancel{\mu}\bar{A} + \dfrac{\nabla(\nabla\cdot\bar{A})}{j\omega\cancel{\mu}\epsilon / \mu} \end{cases}$$

μ ?

H.W. will show that

$|\bar{R}| \gg \lambda$
$$\begin{cases} \bar{E} \longrightarrow -j\omega\mu\bar{A}_T & (\bar{A}_T = \text{tranverse component of } \bar{A}) \\[2ex] \bar{A}_T \text{ is } \perp \text{ to } \bar{R} \\[2ex] \bar{H} = \dfrac{1}{\eta}\hat{k} \times E = \dfrac{1}{\eta}\dfrac{\bar{k}}{|k|} \times \bar{E} = \dfrac{1}{\eta}\bar{u}_k \times \bar{E} \end{cases}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

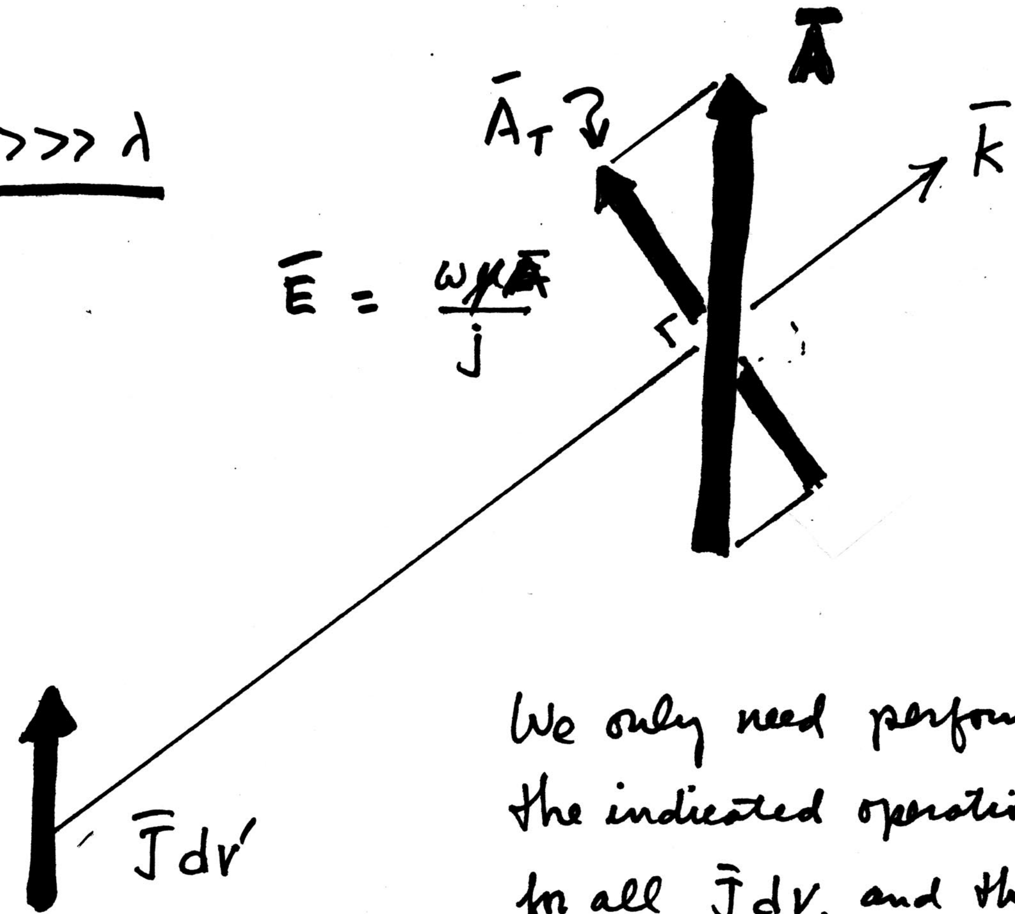


Three points equally distant from source $\bar{J}dv'$

$$d\bar{A}_1 = d\bar{A}_2 = d\bar{A}_3 \quad (!)$$

$$d\bar{A}_{T_1} = 0, \quad |d\bar{A}_{T_2}| = \left|\frac{d\bar{A}_2}{\sqrt{2}}\right|, \quad d\bar{A}_{T_3} = d\bar{A}_3$$

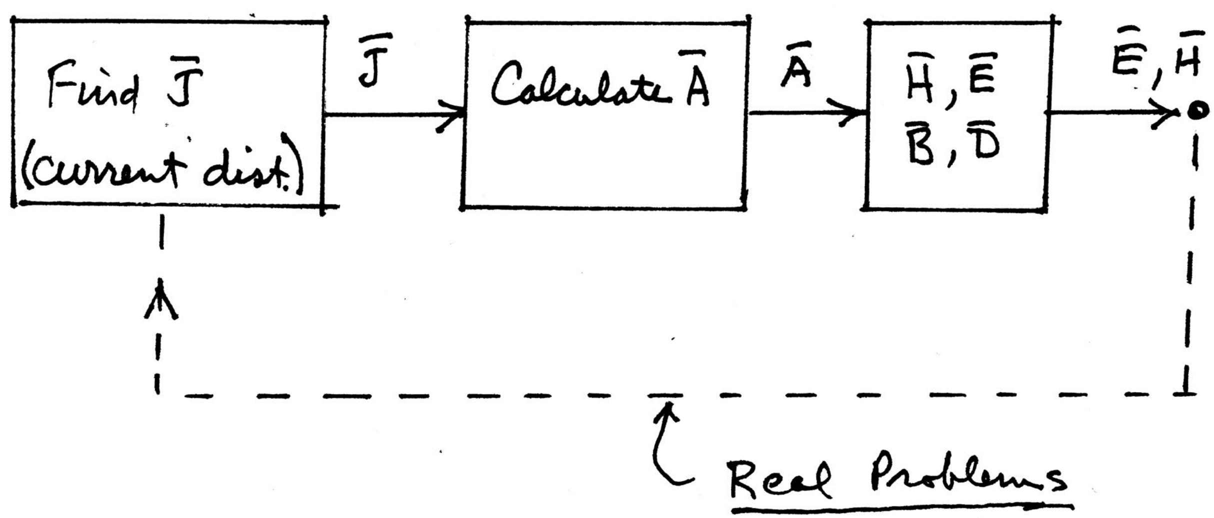
$R \ggg \lambda$



$\bar{E} = \dfrac{\omega \mu \bar{A}}{j}$

We only need perform the indicated operation for all $\bar{J}dv$, and then add up the results.

Procedure, works in principle for all cases



General approach in antennas is to assume $\bar{J}$ (!!!)

Numerical methods now routinely permit sol'ns for $\bar{J}$ when necessary.

In EE 252 we will usually assume $\bar{J}$

$$\bar{A} = \int_{\text{volume}} \frac{\mu[\bar{J}]}{4\pi R}\, dv' \qquad \underline{\text{Radiation Integral}}$$

shows that all radiation can be described in terms of $\bar{J}$, if $\bar{J}$ is known.