Application of Radiation Integral to Simple Antennas

Kraus'
"short" dipole

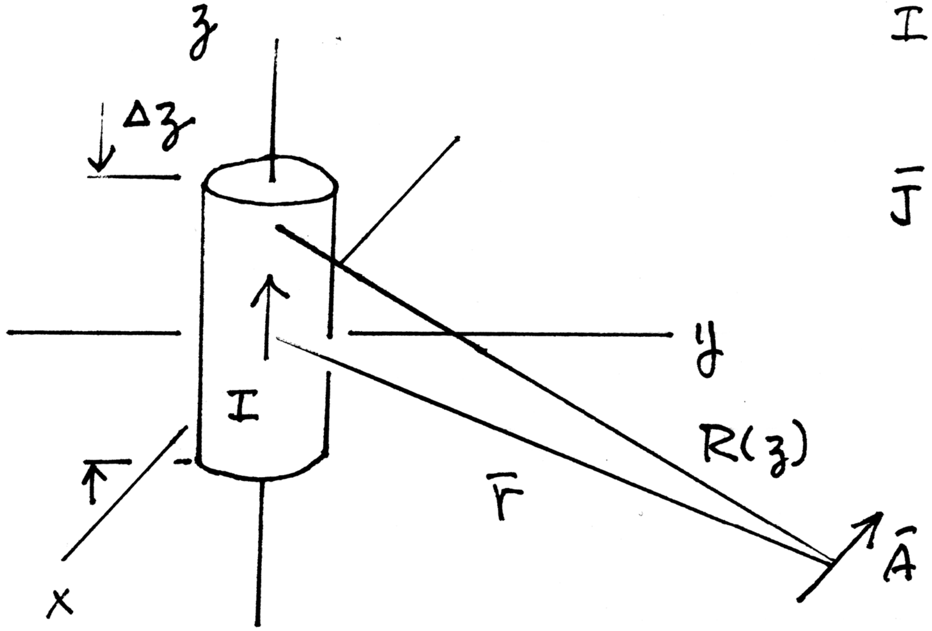
Dipoles – three types

Differ by $\overline{J}$

- Ideal, or ~~infinitesimal~~ ($<< \lambda$)
- Short, with $I=0$ at ends.
- Fractional wavelength, or $f-\lambda$

This lets us examine field structure and segue to larger systems.

Ideal Dipole: fictitious, useful as part of a larger system



$$I = I_0 e^{j\omega t}$$

$$\bar{J} = \bar{a}_z I_0 \delta(\rho),$$

$$-\frac{\Delta z}{2} \le z \le \frac{\Delta z}{2}$$

All current is considered to be concentrated on z axis, between $-\frac{\Delta z}{2}$ and $\frac{\Delta z}{2}$ /...

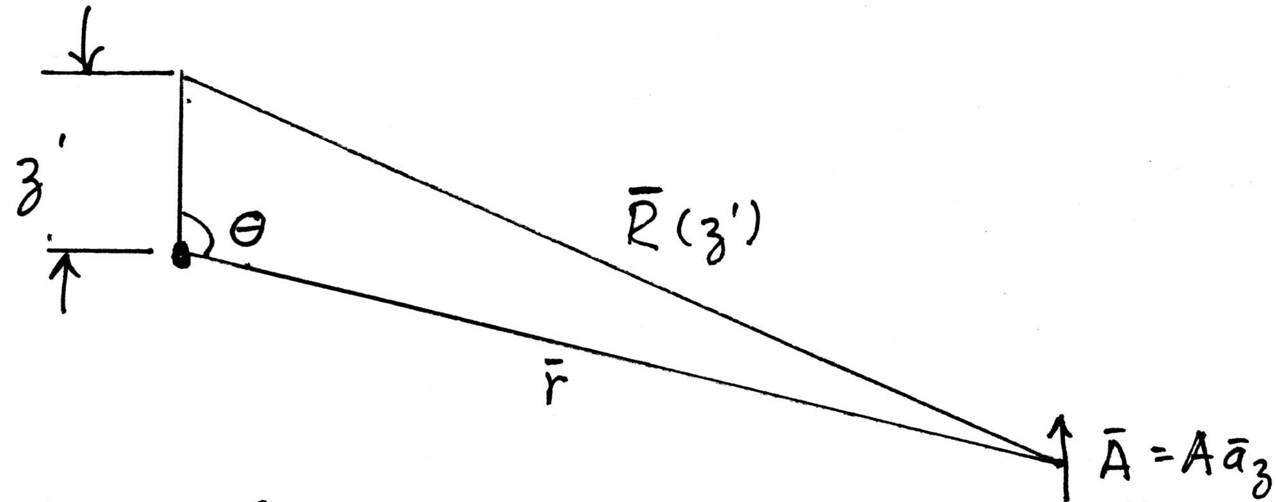
/ ... Why is above called a dipole?

$$\bar{A} = \bar{a}_z \, I_0 \int_{-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} \mu \frac{e^{-jkR}}{4\pi R} \, dz' \quad , \quad \bar{A} \parallel \text{to } z$$

NOTE: TIME VARIATION IS IMPLICIT IN "I"

How to integrate?

/ ...



Law of Cosines

$$R^2 = r^2 + z'^2 - 2rz'\cos\theta$$

$$= r^2\left(1 + (z'/r)^2 - \frac{2z'}{r}\cos\theta\right)$$

$$R(z') \simeq r\left(1 + \frac{1}{2}\left(\frac{z'}{r}\right)^2 - (z'/r)\cos\theta + \cdots\right)$$

/...

/ ...

$$R(z') = r\left(1 - z'/r \cos\theta + \tfrac{1}{2}(z'/r)^2 - \cdots\right)$$

If $\Delta z \ll \lambda, r$

$$\bar{A} \simeq \bar{a}_z \frac{\mu I e^{-jkr}}{4\pi r} \Delta z$$

this is zero order in z'

$$\bar{H} = \frac{\nabla \times \bar{A}}{\mu} = ?$$

/ ...

/... $A_R, A_\theta, A_\phi = ?$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\bar{A} = A_z \cos\theta\, \bar{a}_r - A_z \sin\theta\, \bar{a}_\theta$$

$$\therefore\ \bar{A} = \frac{\mu I_0 \Delta z}{4\pi r} e^{-jkr} \left[\cos\theta\, \bar{a}_r - \sin\theta\, \bar{a}_\theta \right]$$

Now we calculate $\bar{H}$ /...

An Aside

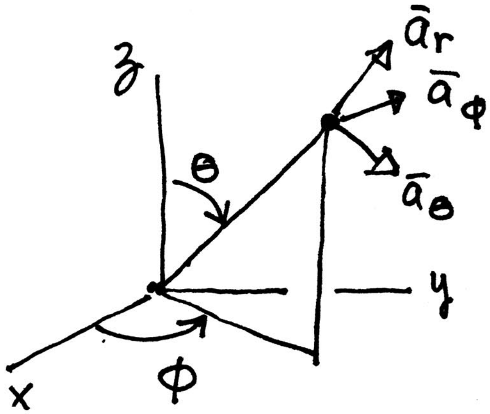
We need a formula . . .

$$\nabla \times \bar{G} \Bigg| = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(G_\phi \sin\theta) - \frac{\partial}{\partial\phi} G_\theta\right]\bar{a}_r$$

spherical coordinates

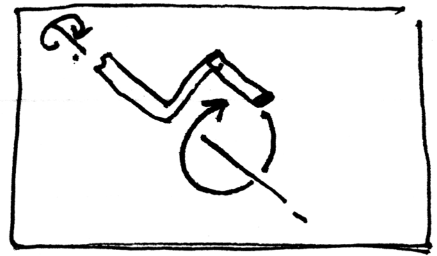
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial G_r}{\partial\phi} - \frac{\partial}{\partial r}\left(r\, G_\phi\right)\right]\bar{a}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\, G_\theta\right) - \frac{\partial G_r}{\partial\theta}\right]\bar{a}_\phi$$



/...

$\bar{H} = ?$; from $\dfrac{\nabla \times \bar{A}}{\mu}$ ...



$$\bar{H} = 0\,\bar{a}_r + 0\,\bar{a}_\theta$$

$$+ \frac{1}{\mu r}\left[+\frac{\partial}{\partial r}\left(r\left[-A_z \sin\theta\right]\right) - \frac{\partial}{\partial\theta}\left(A_z \cos\theta\right)\right]\bar{a}_\phi$$

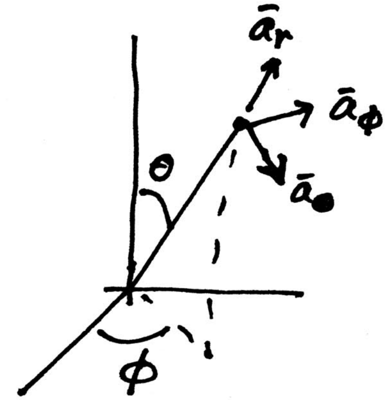
$$= \frac{1}{\mu r}\left[\frac{\partial}{\partial r}\left(-\frac{I\Delta z}{4\pi} e^{-jkr}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{I\Delta z}{4\pi r} e^{-jkr}\cos\theta\right)\right]\bar{a}_\phi$$

$$= \frac{1}{\mu r}\left[j\frac{kI\Delta z}{4\pi} e^{-jkr}\sin\theta + \frac{I\Delta z}{4\pi r} e^{-jkr}\sin\theta\right]\bar{a}_\phi$$

/...

$$\bar{H} = j\frac{k I \Delta z \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr}\right]e^{-jkr}\,\bar{a}_\phi$$

note that $H_r = H_\theta = 0$



$$\vec{E} = \frac{\nabla \times \bar{H}}{j\omega\epsilon_0}$$ in charge free, current free region

$$\nabla \times \bar{H} = \underbrace{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi \sin\theta)\right]\bar{a}_r}_{\mathrm{I}} - \underbrace{\frac{1}{r}\left[\frac{\partial}{\partial r}(r H_\phi)\right]\bar{a}_\theta}_{\mathrm{II}} + 0$$

/...

$$\text{"}I\text{"} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left[\frac{jkI\Delta z}{4\pi r}\left(1+\frac{1}{jkr}\right)e^{-jkr}\sin^2\theta\right]\right]$$

$$= \frac{jkI\Delta z}{2\pi r^2}\left(1+\frac{1}{jkr}\right)e^{-jkr}\cos\theta$$

$\frac{\partial}{\partial\theta}\sin^2\theta = 2\sin\theta\cos\theta$ is only dif. req'd

/...

/ ...

$$"\Pi" = -\frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{jkI\Delta z}{4\pi}\left(1+\frac{1}{jkr}\right)e^{-jkr}\sin\theta\right)\right]$$

$$= -\frac{1}{r}\, j\frac{kI\Delta z}{4\pi}\left[-jk\left(1+\frac{1}{jkr}\right)-\frac{1}{jkr^2}\right]e^{-jkr}\sin\theta$$

$$= (jk)^2\,\frac{I\Delta z}{4\pi r}\left[1+\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]e^{-jkr}\sin\theta$$

$$E = \frac{\nabla\times\bar{H}}{j\omega\epsilon_0} = \ ?$$ (in free space)

/ ...

$$\frac{jk}{j\omega\epsilon_0} = \frac{\omega\sqrt{\mu_0\epsilon_0}}{\omega\epsilon_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \eta_0$$ "<u>Impedance of free Space</u>"

$$\bar{E} = \eta_0 \frac{I\Delta z}{2\pi r^2}\left[1 + \frac{1}{jkr}\right] e^{-jkr} \cos\theta\, \bar{a}_r$$

$$+ j\frac{k\eta I\Delta z}{4\pi r}\left[1 + \frac{1}{jkr} + \frac{1}{(jkr)^2}\right] e^{-jkr} \sin\theta\, \bar{a}_\theta$$

So

$$\boxed{\begin{aligned} E_r &= \eta \frac{I\Delta z}{2\pi r^2}\left[1 + \frac{1}{jkr}\right] e^{-jkr} \cos\theta \\ E_\theta &= jk\eta \frac{I\Delta z}{4\pi r}\left[1 + \frac{1}{jkr} + \frac{1}{(jkr)^2}\right] e^{-jkr} \sin\theta \end{aligned}}$$

$$\boxed{\begin{aligned} E_\phi &= 0 \\ H_\theta &= 0 \\ H_r &= 0 \end{aligned}}$$

Note that in these solutions the radius "r" never appears alone inside the brackets or in the exponent, and that only the leading factors contain a single uncompensated "r".
We can rewrite these equations with a slight change as

$$\bar{H} = j \frac{kI\Delta z}{4\pi r} \left[1 + \frac{1}{jkr} \right] e^{-jkr} \bar{a}_\phi \sin\theta$$

$$= j \frac{I(k\Delta z)}{2\lambda(kr)} \left[1 + \frac{1}{jkr} \right] e^{-jkr} \bar{a}_\phi \sin\theta$$

etc.

From the above we see that in a system with all dimensions measured in (or scaled by) wavelengths, field strength varies inversely with λ, or directly with frequency.

Also note that in obtaining our solution we assumed that $\Delta z \ll r, \lambda$ (this is required by the first order approximation to the law of cosines, above). <u>Thus you cannot approach the dipole with these solutions</u>!!