# EE269
# Signal Processing for Machine Learning

Lecture 2

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# Basic definitions: (Digital) Signal Processing

- **Digital** The origin of the word digital is *digitus*, Latin for finger. Computers store information using only lists or sequences of numbers.
- **Signal** A signal is a function of one or more variables and contains information about the behavior or nature of some phenomenon.
- **Processing** Algorithms for manipulating digital signals in order to extract information.

# Basic notation

- Real or complex valued discrete signals

  $x[n] \;:\; \mathbb{Z} \to \mathbb{C}$ or $x[n] \;:\; \mathbb{Z} \to \mathbb{R}$

  $n \in \mathbb{Z}$ integer index, e.g., discrete time

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- Example: triangle signal $x[n] = ((n+5) \ mod \ 11) - 5$

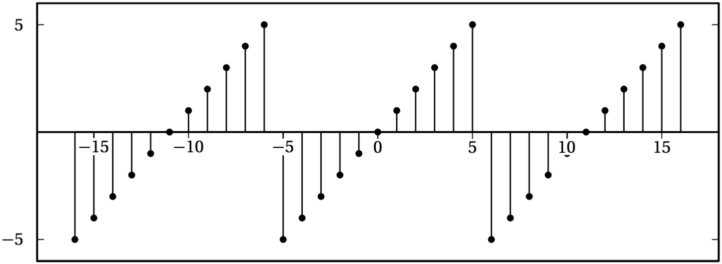



**Figure 2.1** Triangular discrete-time wave.

# Basic notation

- Example: $x[n]$ = Average Dow-Jones index in year $n$

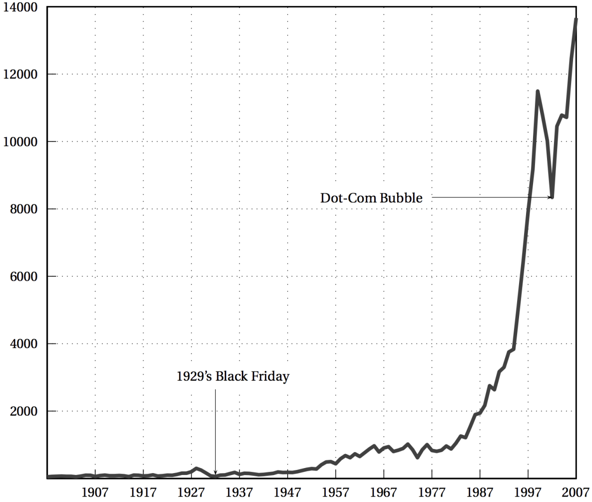



**Figure 2.3** The Dow-Jones industrial index.

# Basic notation

- Continuous signal $x(t)$

  $x(t) \ : \ \mathbb{R} \to \mathbb{C}$ or $x(t) \ : \ \mathbb{R} \to \mathbb{R}$

  $t \in \mathbb{R}$ continuous index, e.g., time

# Basic notation

- Example: $x(t)$ temperature at time $t$

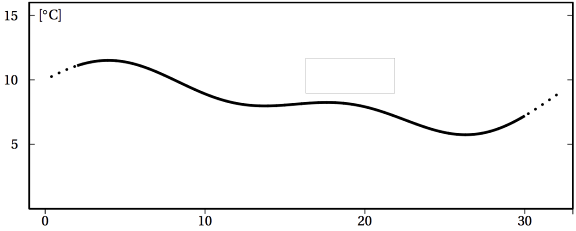



**Figure 1.3** Temperature "function" in a continuous-time world model.

# Basic notation

- Digital signal $x_{\text{dig}}[n]$

  $x_{\text{dig}}[n] \; : \; \mathbb{Z} \to \mathbb{Z}$

  $n \in \mathbb{Z}$ discrete index

# Quantizing discrete signals to digital signals

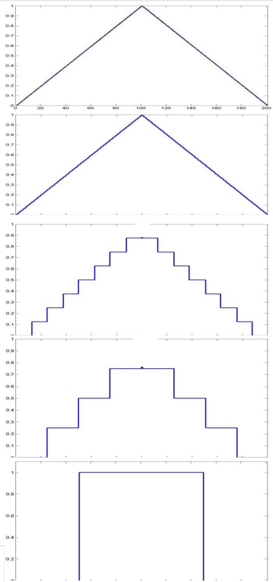
The original signal

8 bit quantization

3 bit quantization

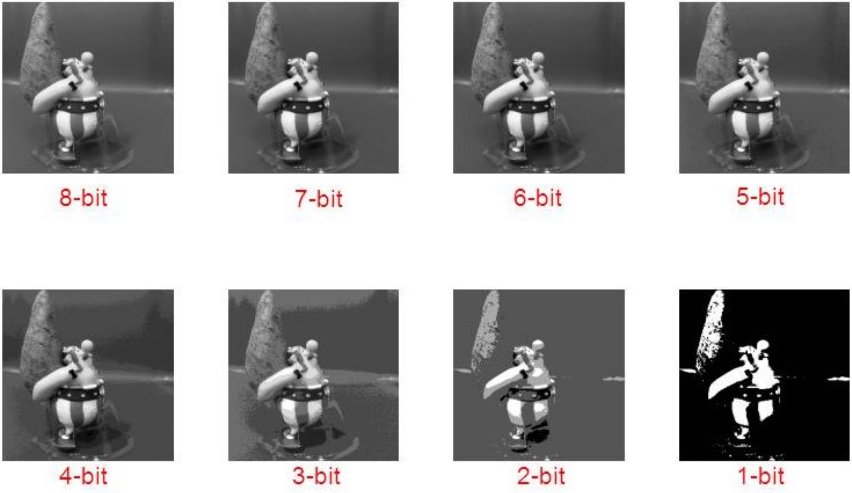
2 bit quantization

1 bit quantization



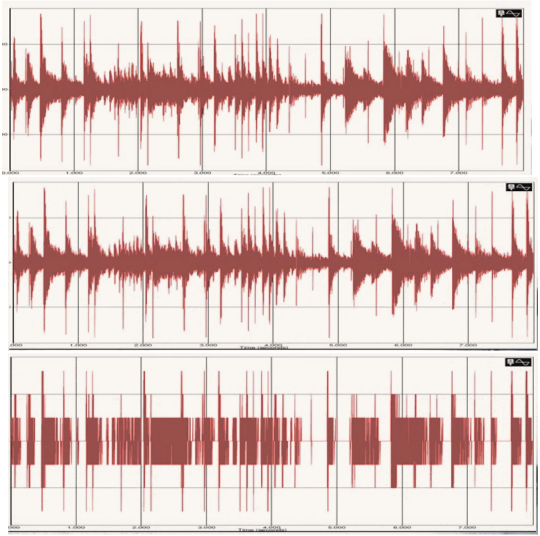
slide credit: B. Raj

# Quantizing images




slide credit: G. Anbarjafari

# Quantizing audio



24 bit

8 bit

3 bit

slide credit: M. Mohan

# Basic notation

- Delta sequence

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

# Basic notation

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$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

- $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$

# Basic notation

- Unit step

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

# Basic notation

- Exponential decay

  $x[n] = a^n u[n]$

  $a \in \mathbb{C}$

  $|a| < 1$

# Basic notation

- Complex exponential $x[n] = e^{j(w_0 n+\phi)}$
  $j = \sqrt{-1}$
  $w_0$ : frequency
  $\phi$ : phase

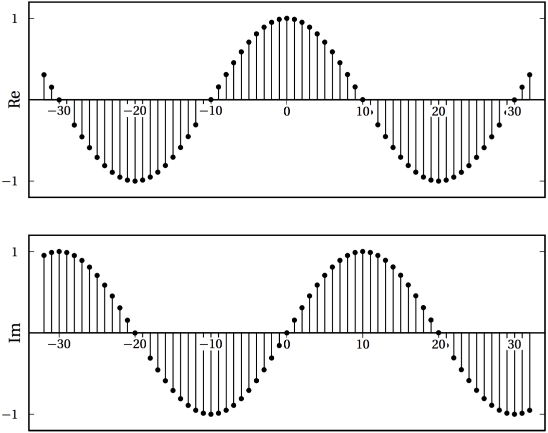
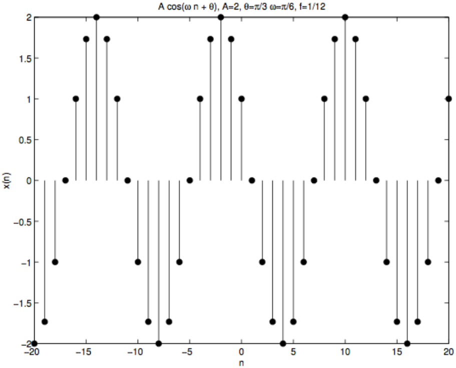



**Figure 2.2** Discrete-time complex exponential $x[n] = e^{j\frac{\pi}{20}n}$ (real and imaginary

Discrete-domain complex exponential signal:

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j(\omega_0 n + \phi)} + \frac{A}{2}e^{-j(\omega_0 n + \phi)} .$$

- $A$ is the amplitude.
- $\phi$ is the phase in radians.
- $n$ is the sample number.
- $\omega_0$ is the frequency in radians per sample.
- Frequency in cycles per sample, $f = \frac{\omega_0}{2\pi}$.



- Periodicity:

  A signal is periodic if there is an $n_0 \in \mathbb{Z}$ such that

$$\tilde{x}[n] = \tilde{x}[n - n_0] \quad \text{for all } n \tag{1}$$

- $\tilde{x}[n]$ : notation for periodic signals

### Theorem (Periodicity of Discrete Sinusoids)

*A discrete-domain or discrete-time sinusoid is periodic if and only if its frequency* $\omega_0$ *is* $\pi$ *times a rational number; that is,*

$$\omega_0 = \frac{M}{N}\pi, \quad M, N \in Z \ .$$

# Sampling a continuous function to get a discrete function

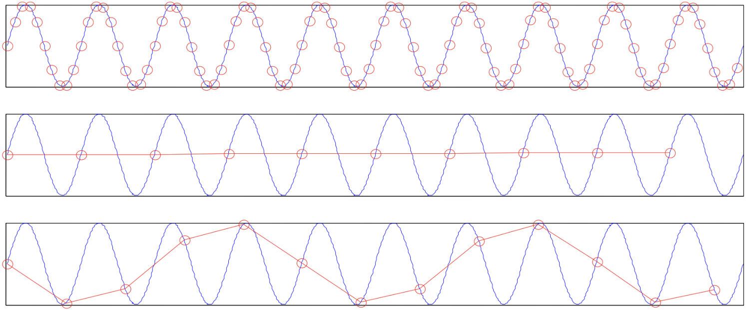
If we sample once every $T$ seconds, then the value of the $n^{th}$ number in the sequence is equal to:

$$x[n] = x_a(nT), \quad -\infty < n < \infty \ .$$

- $T$ is called the **sampling period**
- $1/T$ is called the **sampling frequency**

# Sampling

- **Nyquist-Shannon Sampling Theorem:** If $x(t)$ contains no frequencies higher than $B$ hertz, it is completely determined by its samples $x[n]$ at a series of points spaced $T = \frac{1}{2B}$ seconds apart.



# Aliasing

- **Nyquist-Shannon Sampling Theorem:** If $x(t)$ contains no frequencies higher than $B$ hertz, it is completely determined by its samples $x[n]$ at a series of points spaced $T = \frac{1}{2B}$ seconds apart.
- Lower sampling rate $\implies$ **aliasing**
- Wagon-wheel effect:

  Human eye sampling rate $T \approx \frac{1}{25}$ seconds

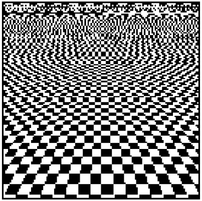
# Sampling a continuous function to get a discrete function

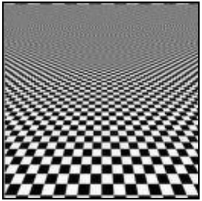
$$x[n] = x_a(nT), \quad -\infty < n < \infty \ .$$

- Example:

  Continuous analog signal $x_a(t) = \cos(2\pi f_0 t)$
- $x[n] = \cos(2\pi f_0 nT)$
- Nyquist-Shannon: $T \leq \frac{1}{2f_0}$
- Periodic iff $\omega_0$ is $\pi$ times a rational number
- $x[n] = \cos(\omega_0 n) \implies$

# Aliasing in images





- Anti-aliasing filters

# Aliasing in images

- NVIDIA: Deep Learning Super Sample

# Energy and Power

- Definition: *Energy* of a discrete-time signal

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \ .$$

- The energy of the signal is finite only if the defined sum converges, in which case we call $x[n]$ *square summable*.

# Energy and Power

- Definition: *Power* of a signal as the ratio of energy over time:

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2 \ .$$

- Finite energy signals (i.e., square summable signals) always have zero power.

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- Definition: *Power* of a signal as the ratio of energy over time:

$$P_x = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2 \ .$$

- Finite energy signals (i.e., square summable signals) always have zero power.
- Question: If a signal is not square summable, can it have finite power?

# Signal Processing and Geometry

- Finite-length Signals: A finite-length discrete-time signal of length $N$ is just a list of $N$ real or complex numbers. This signal is equivalent to a length-$N$ vector.
- We will use two notations equivalently:

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} x_0 & x_1 & \dots & x_{N-1} \end{bmatrix}^T$$

(where the $^T$ denotes transpose) as well as

$$x[n], \quad n = 0, \dots, N-1 \; .$$

# Geometry in $\mathbb{C}^N$

- **Zero vector: $\mathbf{0}$**

  All the other vectors are defined relative to zero.

- **Inner Product**: The inner product between two vectors $x, y \in \mathbb{C}^N$ is defined as

$$\langle x, y \rangle = \sum_{k=0}^{N-1} x_k^* y_k$$

  We say that $x$ and $y$ are *orthogonal*, or $x \perp y$, when their inner product is zero: $\langle x, y \rangle = 0$.

# Norm

- The norm of a vector $x \in \mathbb{C}^N$ is defined as

$$\|x\| = \sqrt{\sum_{k=0}^{N-1} |x_k|^2} = \langle x, x \rangle^{1/2} .$$

  This is called the 2-norm or $\ell_2$-norm and denoted by $\|x\|_2$.
- Gives the length of the vector in $\mathbb{R}^2$, i.e., the distance from zero.

# Norm in two dimensions

- For two dimensional real vectors, $\mathbb{R}^2$, the definitions of inner product and norm are related by the cosine of the angle between the two vectors.

$$\langle x, y \rangle = x_0 y_0 + x_1 y_1 = \|x\| \|y\| \cos\theta \ .$$

- When the angle $\theta = \pi/2$, the inner product is zero and the vectors are orthogonal.

- An important link between the inner product and norms is the Cauchy-Schwarz inequality:

Theorem (Cauchy-Schwarz)

$$|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2 \ .$$

- Optimization proof:

# Basis

- A **basis** for a class of signals is a collection of $M$ signals in the class that have the property that *any other signal in that class* can be written as a weighted sum of those signals.

- Suppose we have the class of signals that are length-$N$, and $x[n]$ is in that class (is of length $N$).
  If $y^{(0)}[n], \ldots, y^{(M-1)}[n]$ are also length-$N$ and are a **basis** for these signals, we know we can find $c[1], \ldots, c[0]$ such that

$$x[n] = \sum_{k=0}^{M-1} c[k] y^{(k)}[n] \; .$$

# Why change basis ?

- If we want to compress, we want $c[k]$ to have more small values or zero values than $x[k]$.
- If we want to classify (e.g., recognize different speakers), we want $c[k]$ to have spikes in different locations for different classes (e.g., different frequencies will have large Fourier coefficients).

- If we want to separate sources (e.g., separate sources of air pollution given measurements across the city), we want $c[k]$ again to have spikes for different $k$ depending on the source (e.g., using a spatial-group sparse basis).
- If we want to reconstruct (e.g., image inside the body from external measurements), we want $c[k]$ to capture the most important aspects of the signal (e.g., outlines of tumors; bases designed for preserving these edges include wavelets and curvelets).

# Standard Basis

- The collection of shifted deltas is a basis, because if we set $c[m] = x[m]$, we get

$$x[n] = \sum_{m=0}^{M-1} c[m]\delta[n-m] \, .$$

The shifted deltas are called *the canonical basis* or *the standard basis*. It turns out this is also an **orthogonal basis**, meaning that all the signals in the basis are orthogonal to one another.

# Other basis examples

- Let's look at some bases for $\mathbb{R}^2$, length-2 real valued signals. The delta basis is $\delta_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\delta_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.
- Another basis is

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

  This is called a Hadamard basis for $\mathbb{R}^2$. Also equal to the Haar wavelet basis (only in $\mathbb{R}^2$)

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  **One way to check if vectors form a basis** is to see if you can write all vectors in the canonical basis as a linear combination of these new basis vectors.
- Another basis is

$$x_0 = \begin{bmatrix}3\\4\end{bmatrix}, \quad x_1 = \begin{bmatrix}3\\-4\end{bmatrix}$$

- **How to check if some vectors form a basis ?**

  For the vector space of length-$N$ signals, if we have $N$ **linearly independent** vectors then they form a basis.

  A collection of $N$ signals $y^{(0)}[n], \ldots, y^{(N-1)}[n]$ is *linearly independent* if the following is true:

  $$\sum_{m=0}^{N-1} \beta_m y^{(m)}[n] = 0 \text{ implies that } \beta_m = 0 \text{ for all } m = 0, \ldots, N-1 \; .$$

  If a set of signals is not linearly independent, we call it *linearly dependent*.

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- Let's use this technique to show our example above is a basis:

$$y^{(0)} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad y^{(1)} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

- *Orthonormal Bases:* If we have a basis $y^{(0)}[n], \ldots, y^{(N-1)}[n]$ where all the signals are mutually orthogonal:

$$\langle y^{(k)}[n], y^{(\ell)}[n] \rangle = 0 \text{ for all } k \neq \ell$$

  and if all the signals in the basis have norm 1:

$$\|y^{(k)}[n]\| = 1 \text{ for all } k = 1, \ldots, N$$

  then we call it an *orthonormal basis*.
- Delta basis is an orthonormal basis

- Example basis

$$x_0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix} .$$

Are these vectors orthogonal?

They are not norm one, but we can scale them to be norm 1 by dividing by the norm of each basis vector.

Then we have an orthonormal basis:

- *Fourier Basis:* An important orthonormal basis for length-$N$ complex signals is the *normalized Fourier basis* defined as:

$$w_m[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nm} \quad \text{for} \quad \begin{array}{c} n = 0, \ldots, N-1 \\ m = 0, \ldots, N-1 \end{array}$$

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- Orthogonal ? Orthonormal ?

  $\langle w_k[n],\, w_r[n] \rangle =$

# How to change basis?

- Suppose we have a signal $x[n]$ in the standard basis. In order to write a signal in a different basis, as long as that domain is an **orthonormal basis**, we do the following:

  1. Take the inner product of your signal with every element from the orthonormal basis. These are called the *expansion coefficients*

  $$\langle y^{(k)}[n], x[n] \rangle$$

  2. Multiply each basis vector by the corresponding inner product and sum all the scaled basis vectors together.

  $$x[n] = \sum_{k=0}^{N-1} \langle y^{(k)}[n], x[n] \rangle y^{(k)}[n]$$

# References


Signal processing for communications, Prandoni and Vetterli

Digital Image Processing, Gonzales and Woods

Pattern Recognition and Machine Learning, Bishop

EECS351 Digital Signal Processing Lecture notes, Laura Balzano, University of Michigan