EE269
Signal Processing for Machine Learning

Wavelets Part II

Instructor : Mert Pilanci

Stanford University

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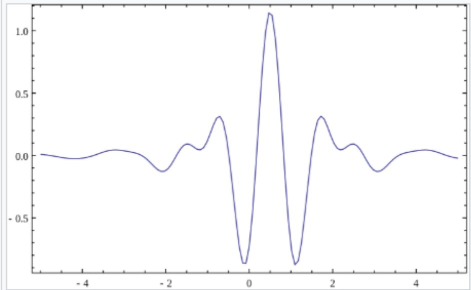
What makes a good wavelet

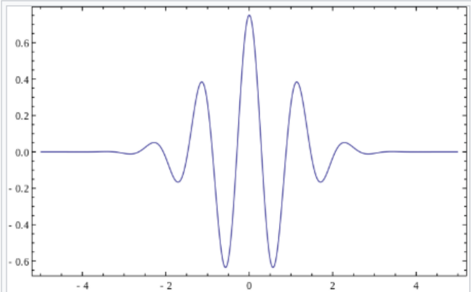
Application specific

- Compact time support vs frequency support
- Smoothness
- Orthogonality

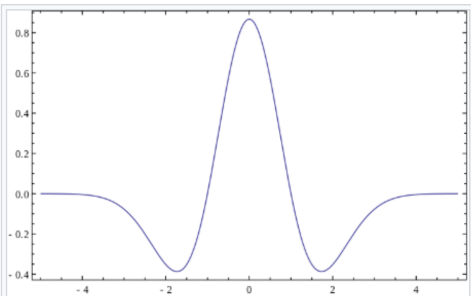
Other Wavelets



Meyer



Morlet

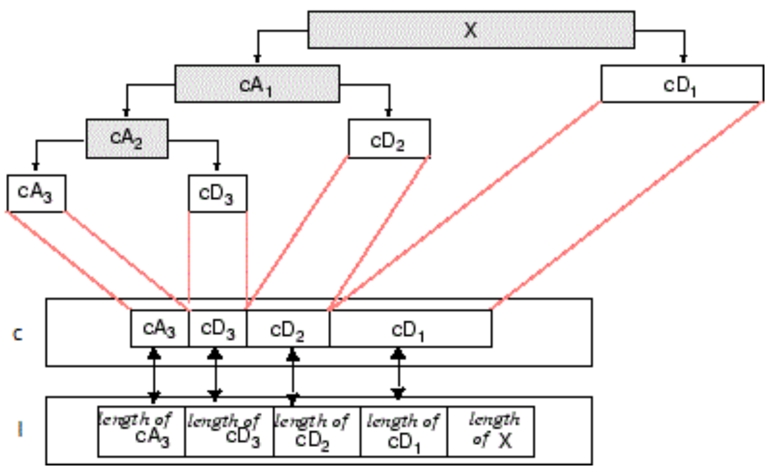


Mexican hat

Other Wavelets

- In MATLAB

  `[c,l] = wavedec(x,n,wname)` returns the wavelet decomposition of the signal x at level n using the wavelet *w*name

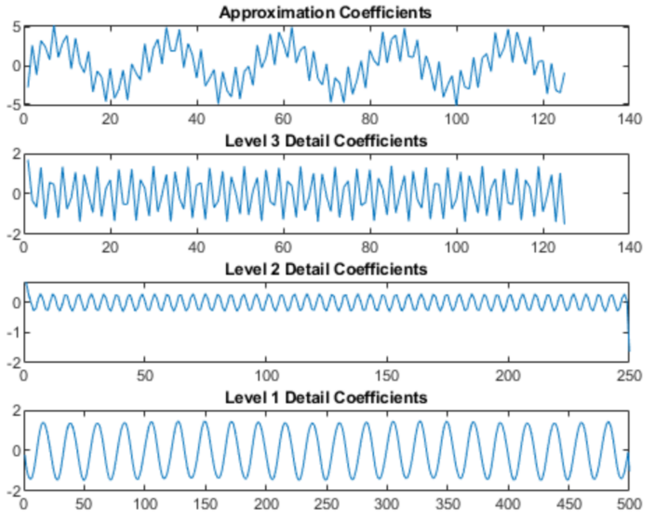


Other Discrete Wavelets

```
[c,l] = wavedec(sumsin,3,'db2');
approx = appcoef(c,l,'db2');
[cd1,cd2,cd3] = detcoef(c,l,[1 2 3]);
```

Plot the coefficients.

```
subplot(4,1,1)
plot(approx)
title('Approximation Coefficients')
subplot(4,1,2)
plot(cd3)
title('Level 3 Detail Coefficients')
subplot(4,1,3)
plot(cd2)
title('Level 2 Detail Coefficients')
subplot(4,1,4)
plot(cd1)
title('Level 1 Detail Coefficients')
```



Fourier vs Wavelet Transforms

- Fourier Transform has convolution theorem and mathematical relationships
- No closed form relations exist for wavelet transforms
- Fourier transform has uniform spectral resolution
- Wavelet transform has adaptive resolution
- 100 Hz resolution at 400 Hz and at 4000 Hz are not the same

Short-time Fourier Transform

- window signal

  e.g. $w[m] = \begin{cases} 0 & m < 0, m \geq L \\ 1 & 0 \leq m \leq L-1 \end{cases}$

- Short Time Fourier Transform (STFT)

$$X[n,k] = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km}, \quad 0 \leq k \leq N-1 \,.$$

Short-time Fourier Transform

- window signal

  e.g. $w[m] = \begin{cases} 0 & m < 0, m \geq L \\ 1 & 0 \leq m \leq L-1 \end{cases}$

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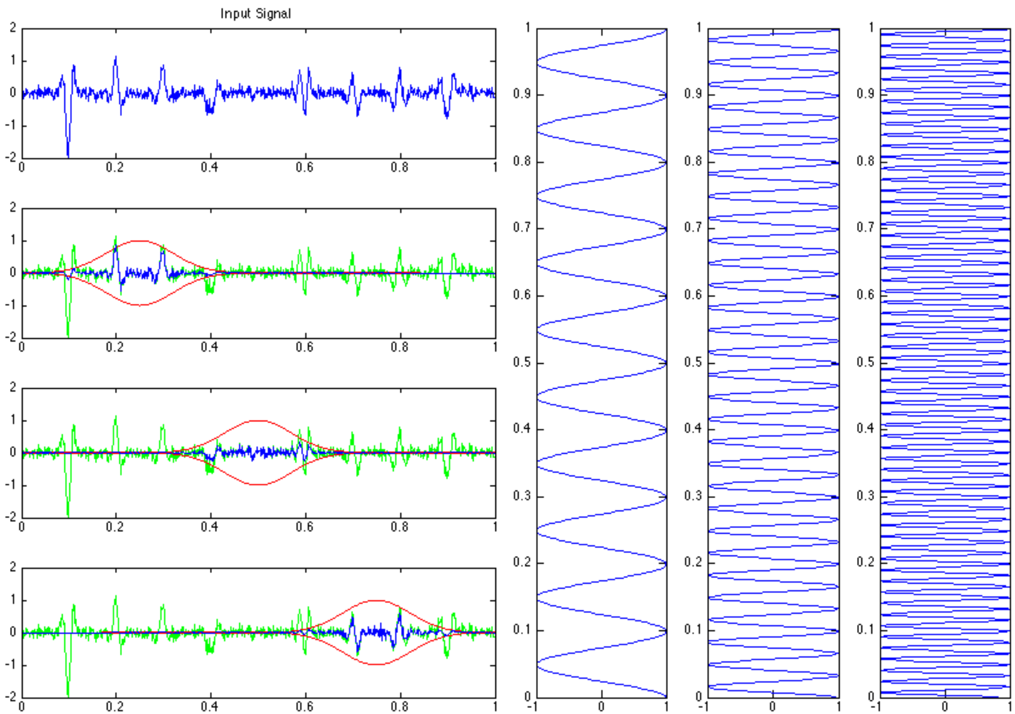
$$X[n,k] = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km}, \quad 0 \leq k \leq N-1 \, .$$

- Continuous Frequency STFT

$$X[n,\lambda] = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j\lambda m},$$

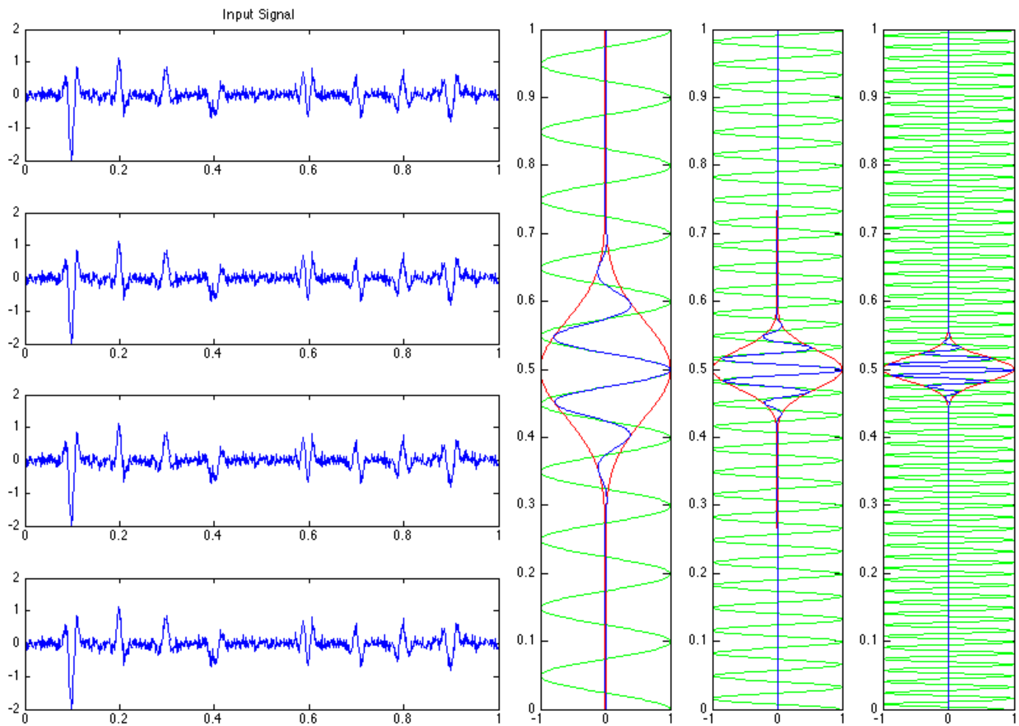
Short-time Fourier Transform



$$X[n,\lambda] = \sum_{m=0}^{L-1} x[n+m]\underbrace{w[m]e^{-j\lambda m}},$$

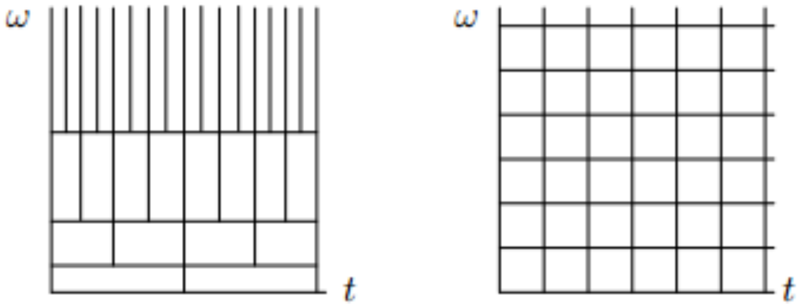
- time-windowed signal

Short-time Fourier Transform vs Wavelet Transform



- windowed signal = windowed complex exponential basis
- STFT has uniform time and frequency resolution
- In contrast, wavelets have adaptive windows:
- short windows for higher frequencies (small scale)
- long windows for lower frequencies (large scale)

Wavelet Transform vs STFT



Wavelet Transform **Short Time Fourier Transform**

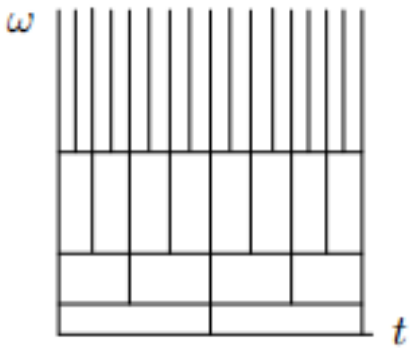
- Wavelet transform analyzes a signal at different frequencies with different resolutions:

  good time resolution and relatively poor frequency resolution at high frequencies

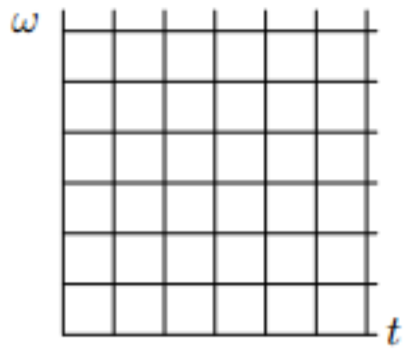
  good frequency resolution and relatively poor time resolution at low frequencies

- Wavelet transform is better for signals with non-periodic and fast transient features (i.e., high frequency content for short duration)

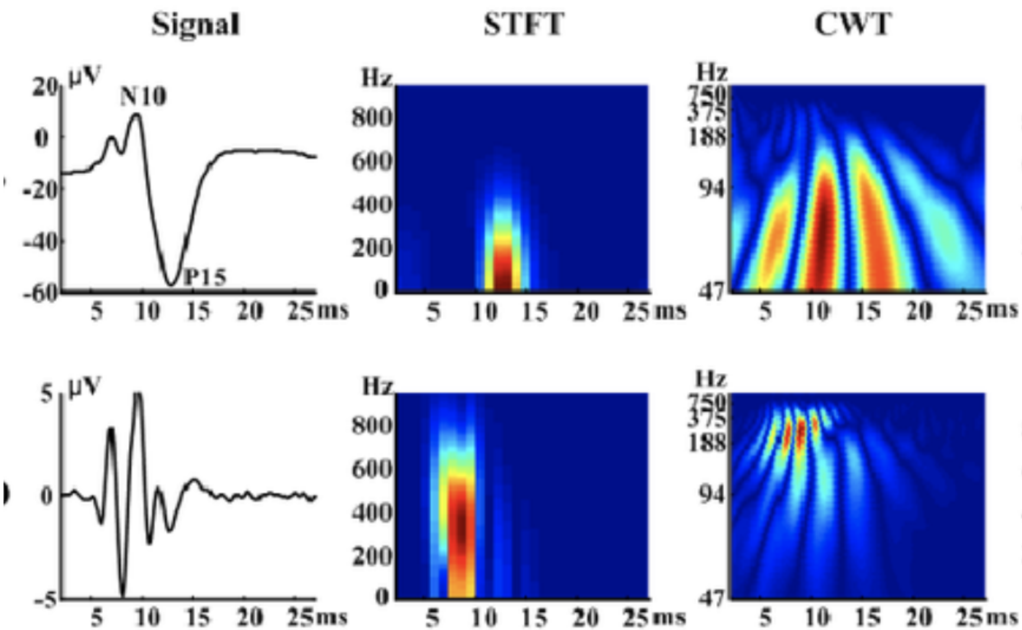
Wavelet Transform vs STFT



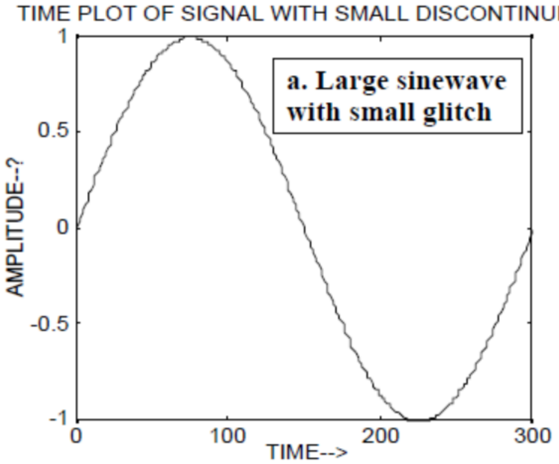
Wavelet Transform

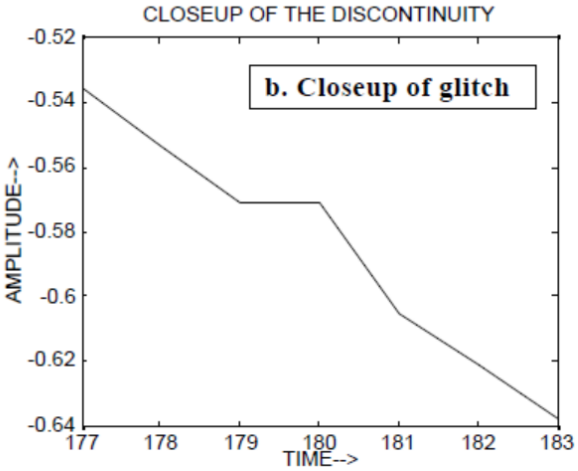


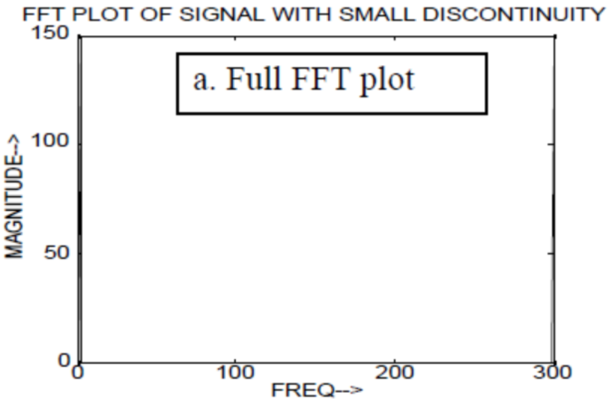
Short Time Fourier Transform

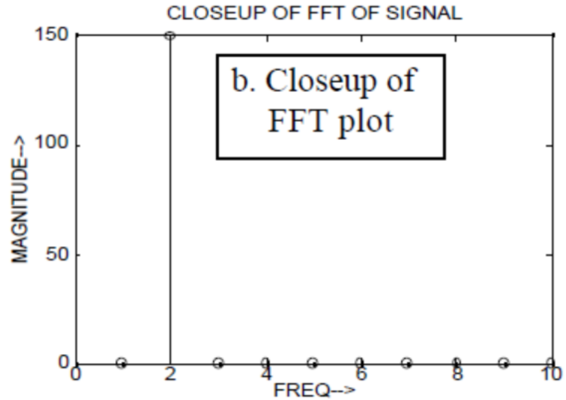


Wavelet Transform vs STFT : Locality

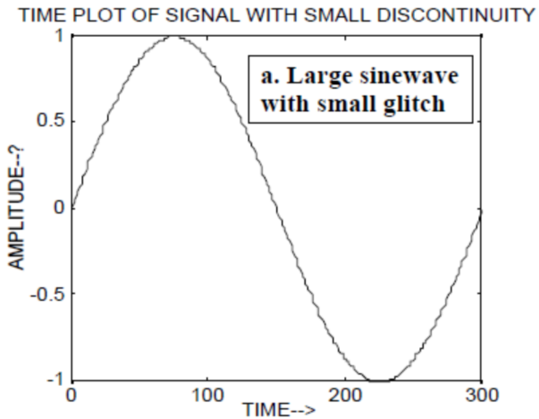


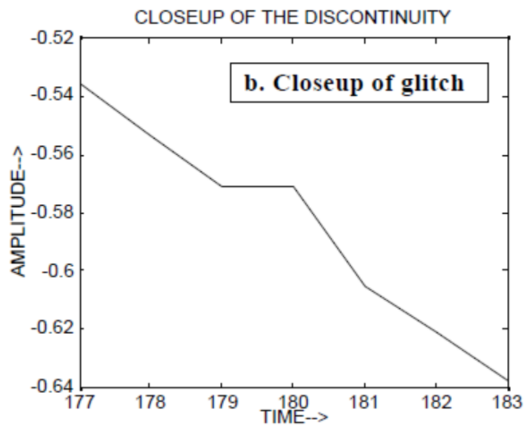


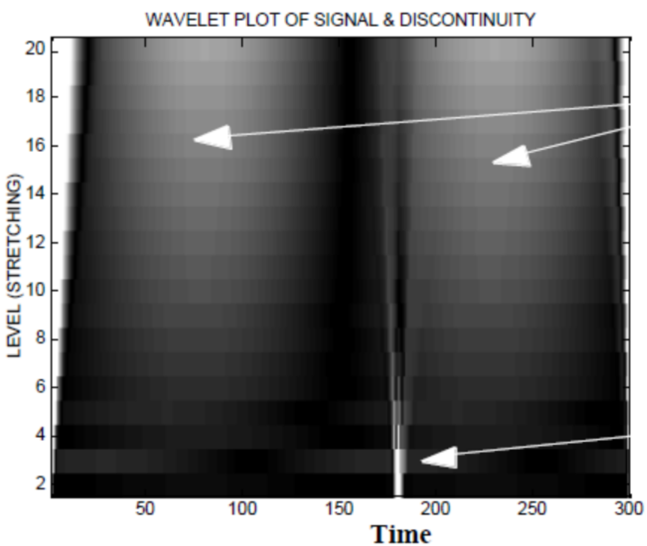

FFT PLOT OF SIGNAL WITH SMALL DISCONTINUITY
a. Full FFT plot
MAGNITUDE-->
FREQ-->
0
50
100
150
0
100
200
300


CLOSEUP OF FFT OF SIGNAL
b. Closeup of FFT plot
MAGNITUDE-->
FREQ-->
0
50
100
150
0
2
4
6
8
10

Wavelet Transform vs STFT : Locality



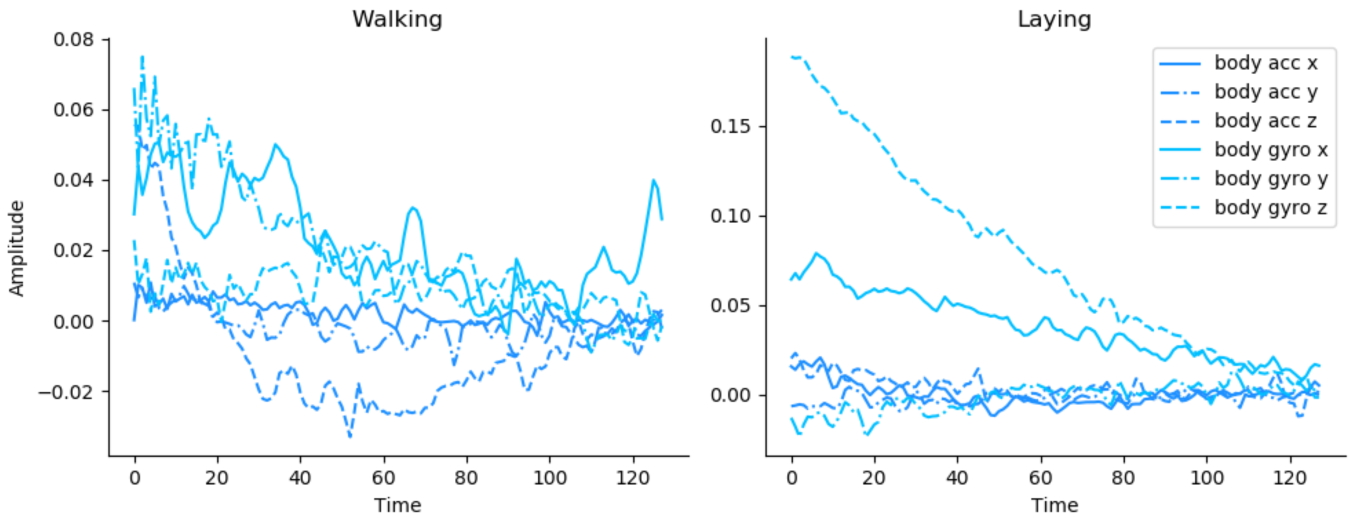




Stretched "low frequency" wavelet compares better to long sinusoidal (wave) signal. It "finds" peaks and valleys.

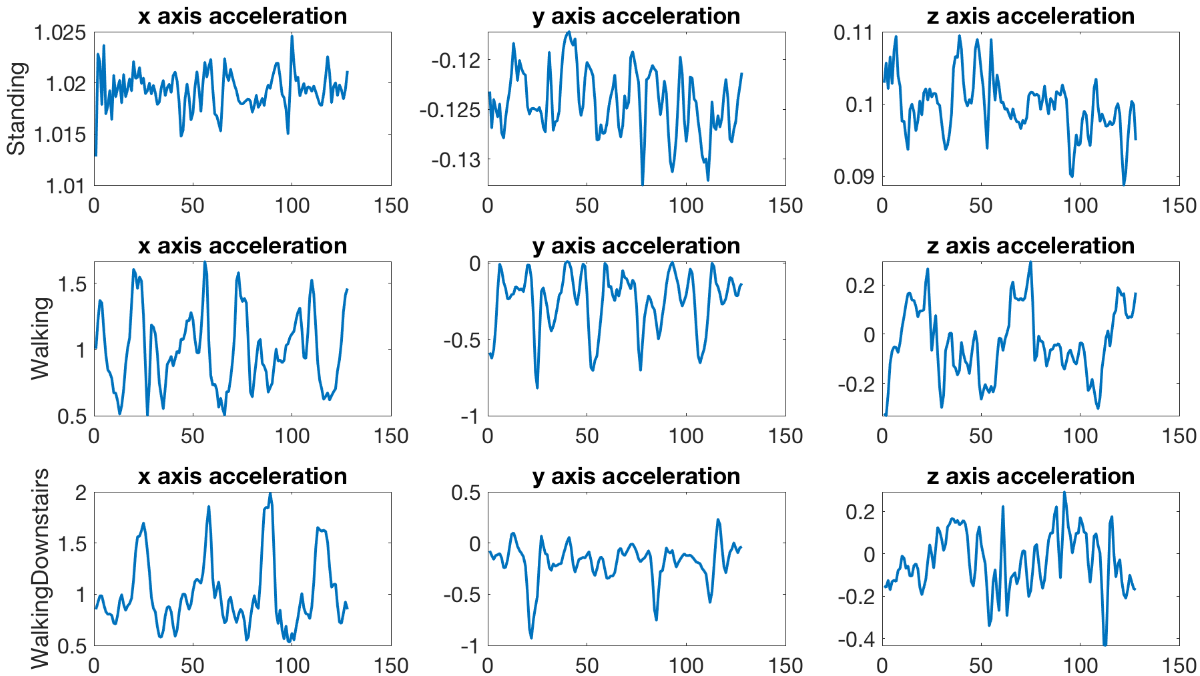
Short "high frequency" wavelet compares well to discontinuity. It "finds" it's location at 180.

Human Activity Recognition (HAR) Dataset

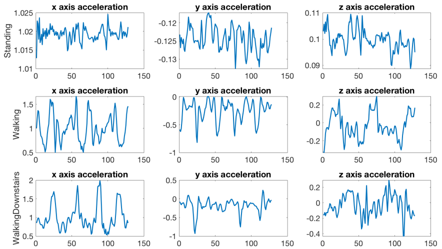


Application

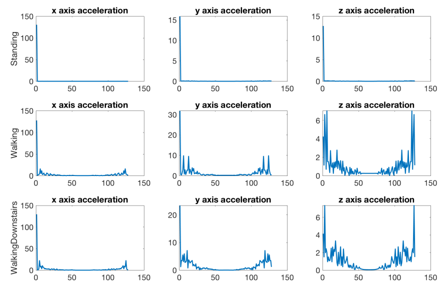
- Human Activity Recognition Using Smartphones Data Set (Reyes-Ortiz et al, 2012)
- Compote DFT of the training signals $X_1[k], X_2[k], \ldots X_m[k]$
  DFT Magnitude $|X_1[k]|, |X_2[k]|, \ldots |X_m[k]|$



Results: training set: 7724 signals, test set: 2575 signals

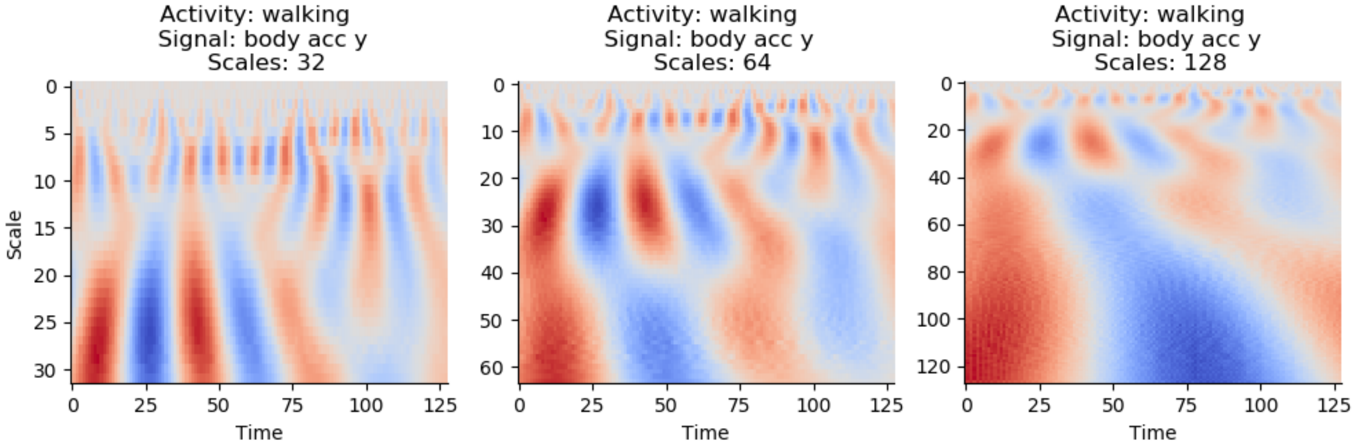


3-Nearest Neighbors, ℓ_2-norm distance on $x[n]$. **Accuracy** : 0.77



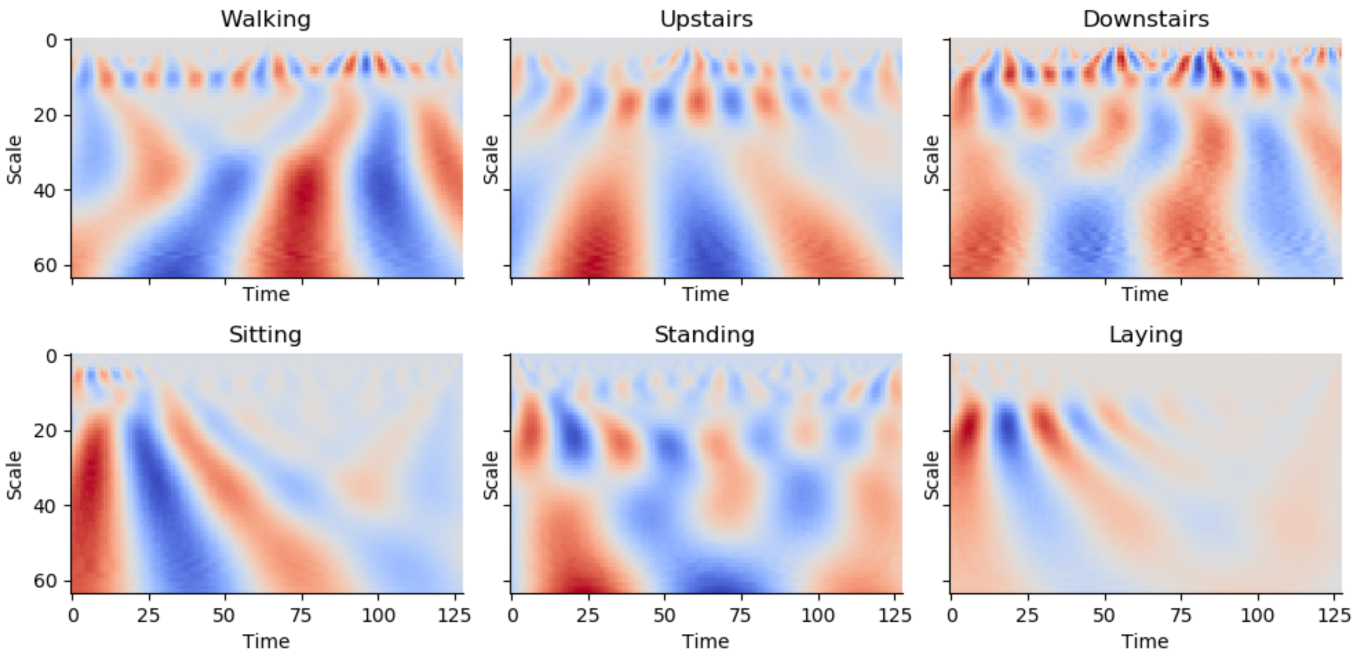
3-Nearest Neighbors, ℓ_2-norm distance on $|X[k]|$. **Accuracy** : 0.85

Continuous Wavelet Transform of HAR signals

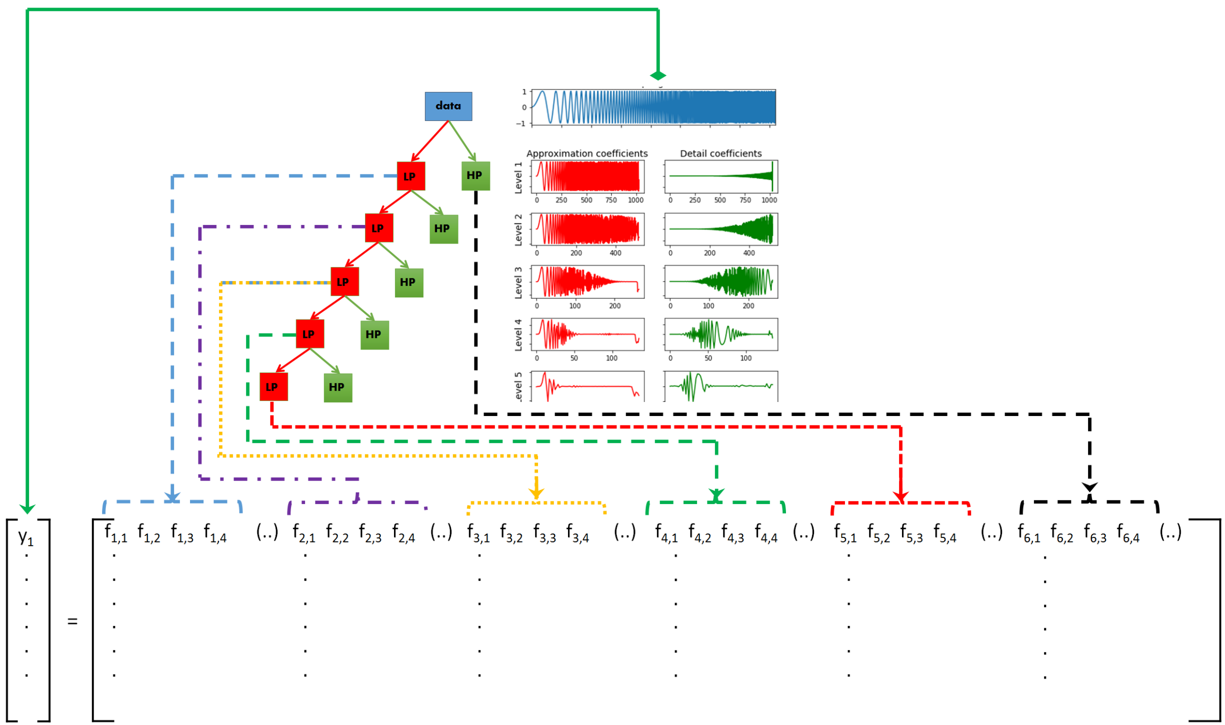


- changing the number of scales

Continuous Wavelet Transform of HAR signals



Wavelet Transform Features



- mean, median
- variance
- zero crossing rate, mean crossing rate
- entropy

Human Activity Recognition dataset

- 3-Nearest Neighbors, ℓ_2-norm distance on $x[n]$.

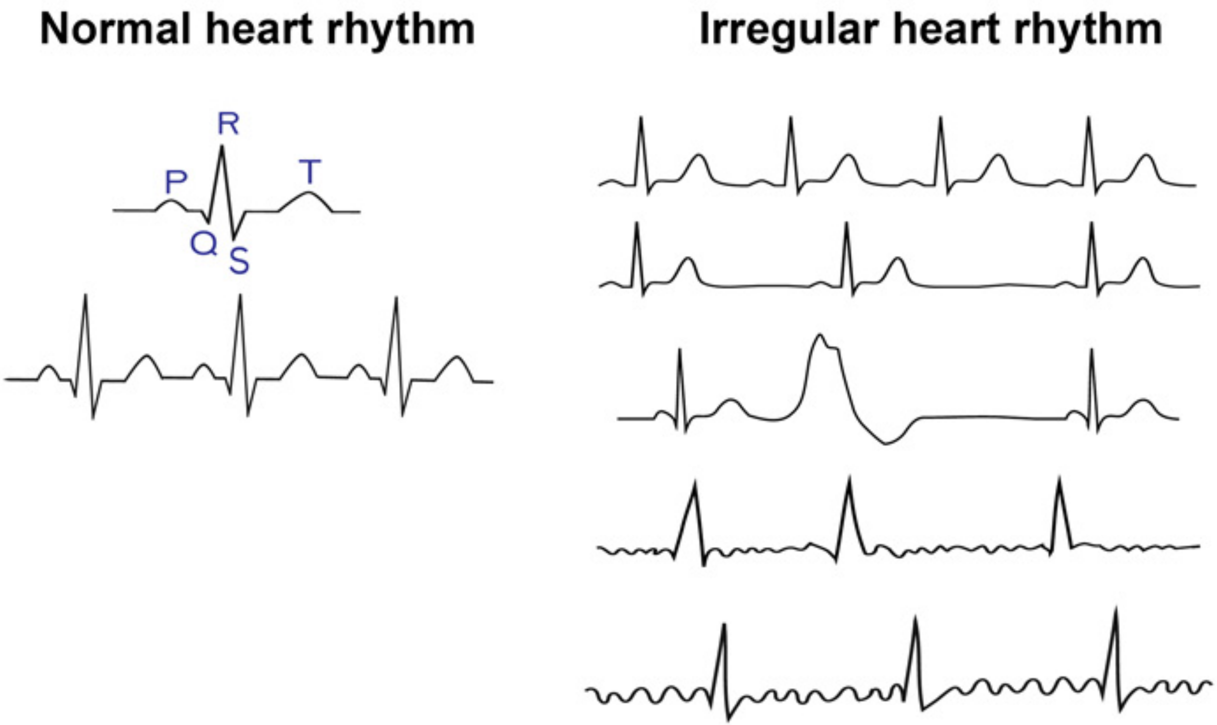
  accuracy : 0.77%
- 3-Nearest Neighbors, ℓ_2-norm distance on $|X[k]|$.

  accuracy : 0.85%
- 1D Convolutional Net

  accuracy : 91%
- Wavelet Transform Features (entropy, zero crossing, simple statistics) $+$ linear classifier

  accuracy : 95%

Application: Arrhythmia Detection



Application: Arrhythmia Detection

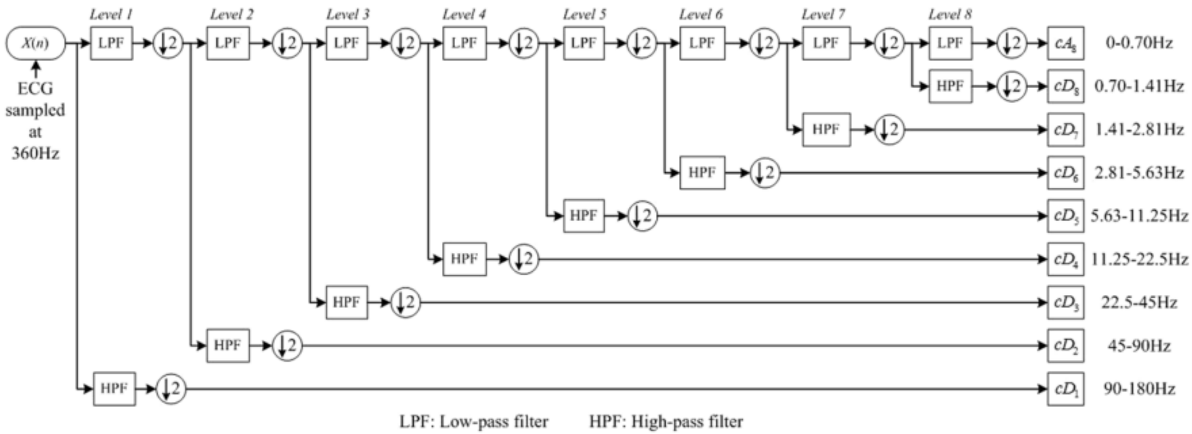


Figure 3. The decomposition process of the 8-level WMRA.

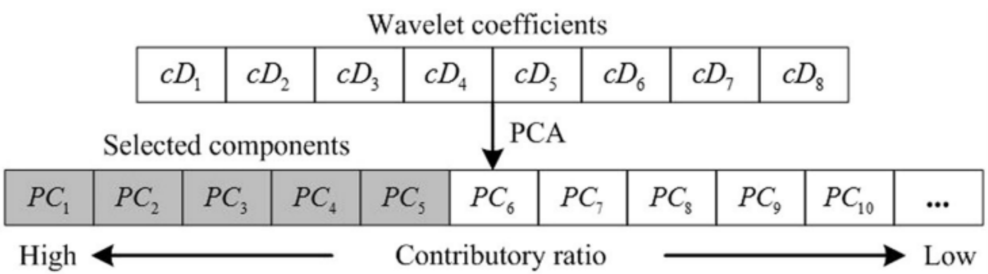


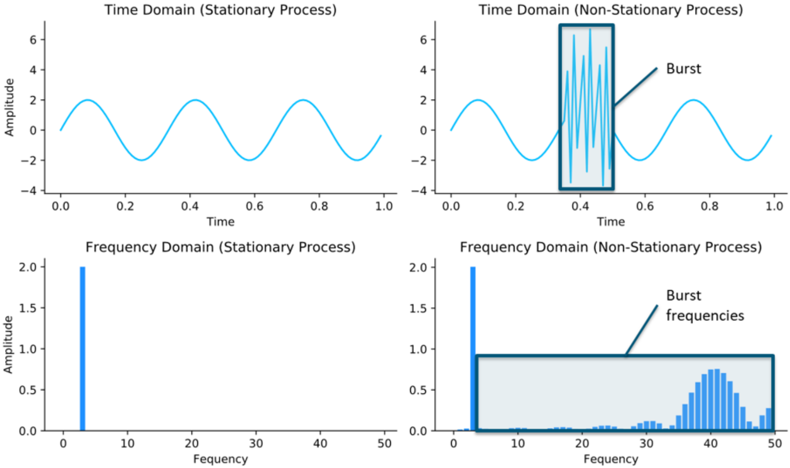
Figure 4. Low-dimensional feature vector generated by PCA using wavelet coefficients.

Application: Arrhythmia Detection

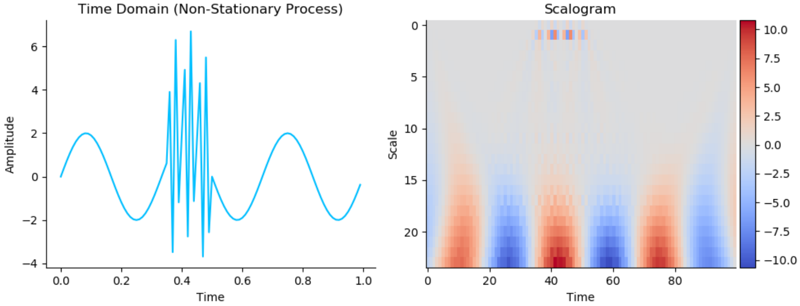
Literatures and feature extraction methods	Feature selection (dimension)	Beat types	Training/test beats	Classifiers	Independent training/test data	k-fold cross validation	*SEN* (%)	*SPE* (%)	*ACC* (%)
Spectral correlation[1]	Yes (88)	5	Totally 6259	SVM	Unknown	10-fold	99.20	99.70	98.60
Wavelet transform, morphological features[5]	No (28)	5	10675/93894	Artificial neural network	No	No	88.60	96.18	97.86
Morphological features[7]	Yes (6)	6	35848/35848	Linear discriminant analysis	No	No	91.19	98.65	94.03
Morphological features[8]	No (13)	3	600/30273	SVM, neural network	No	No	98.52	99.19	97.14
Time domain features[9]	No (9)	6	42427/14142	Decision tree	No	No	97.50	99.80	99.51
Morphological features[10]	No (16)	3	15509/8081	SVM, neural network	Yes	No	92.82	93.74	92.85
Morphological features[11]	No (8)	5	12570/12570	Regression neural network	No	No	85.50	99.40	99.40
Fourier transform, wavelet package[14]	Yes (70)	16	3345/2542	*k*-NN	No	No	85.59	99.56	93.59
Wavelet transform, cosine transform[15]	Yes (18)	4	720/360	SVM	Unknown	No	98.60	95.50	96.50
Wavelet transform[16]	Yes (24)	5	900/900	SVM, genetic algorithm	No	No	98.50	99.69	98.80
Higher order spectral[17]	No (7)	5	330/500	SVM	Unknown	No	90.00	87.93	85.79
Wavelet transform[18]	Yes (20)	4	360/360	SVM	Unknown	No	98.62	99.54	98.61
Temporal and spectral features[21]	Yes (15)	6	1440/720	SVM	No	No	97.60	93.80	95.20
Temporal and spectral features[22]	Yes (13)	8	Totally 17857	SVM	No	5-fold	95.00	99.00	98.60
Higher order statistics, wavelet packet[27]	Yes (28)	5	3345/2542	*k*-NN	Yes	No	89.80	97.80	—
Hilbert-Huang transform[32]	Yes (18)	6	10700/10700	SVM	No	No	98.64	99.77	99.51
Wavelet transform[46]	Yes (18)	5	Totally 101352	SVM	Yes	44-fold	—	—	86.40
		16	24100/86009		No	No	99.32	—	99.01
Approximate entropy, wavelet packet[47]	Yes (9)	5	145/145	SVM, PNN	Unknown	No	98.70	99.70	98.60
Non-linear and center-clipping transform[48]	No (5)	5	13640/13640	Wavelet neural network	No	No	98.78	99.70	98.78
Eigenvector method[49]	Yes (12)	4	360/360	Recurrent neural network	Unknown	No	98.89	99.25	98.06
Higher order statistics[50]	No (24)	5	4000/14299	RBF neural network	No	No	92.93	98.52	95.18
Geometrical features[51]	No (18)	7	4035/3150	SVM, *k*-NN, BPNN	No	No	97.52	99.65	98.06
Wavelet transform, morphological features[52]	Yes (8)	3	50928/49636	Linear discriminant analysis	Yes	No	80.00	—	94.00
Wavelet transform, linear prediction model[53]	No (12)	3	50554/49273	Linear discriminant analysis	Unknown	No	86.50	—	86.50
Cross correlation[54]	No (30)	3	41961/51285	Artificial neural network	Unknown	No	97.49	—	95.24
WMRA [This work]	**Yes (12)**	**6**	**Totally 107049**	**SVM**	**Yes**	**10-fold**	**44.40**	**88.88**	**81.47**
					No		**99.09**	**99.82**	**99.70**

Q. Qin et al., Nature Scientific Reports

Limitations of the Fourier Transform

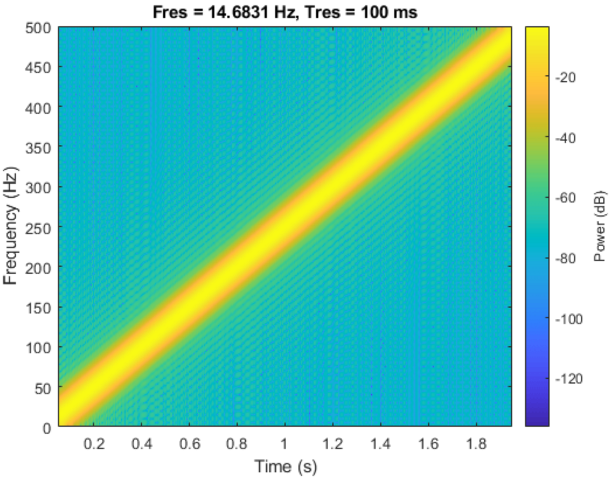


Non-stationary signals



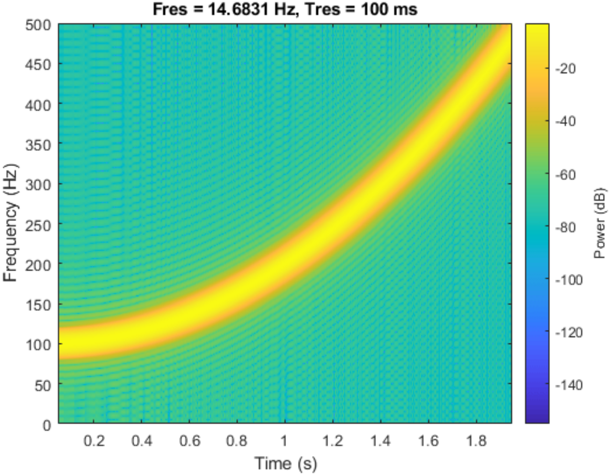
Chirp Signals

- linear chirp $x(t) = \sin(2\pi(ct^2 + f_0 t))$



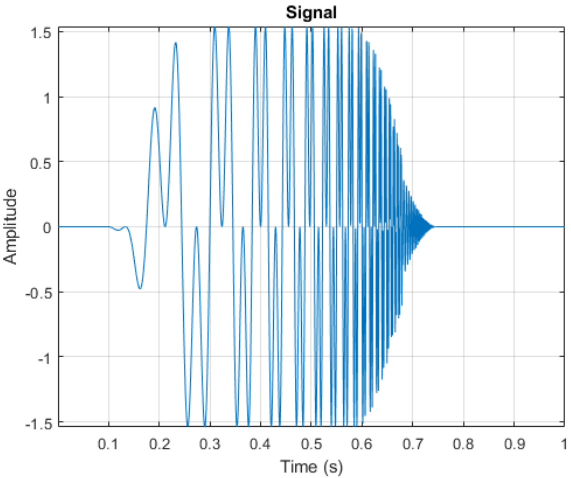
Chirp Signals

- quadratic chirp $x(t) = \sin(2\pi(ct^3 + dt^2 + f_0 t))$

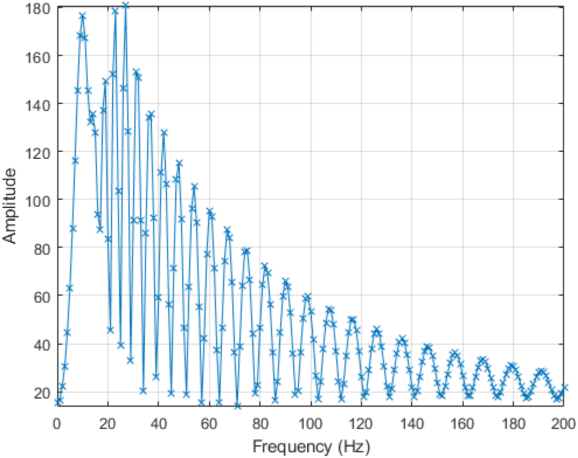


Resolving Signal Components

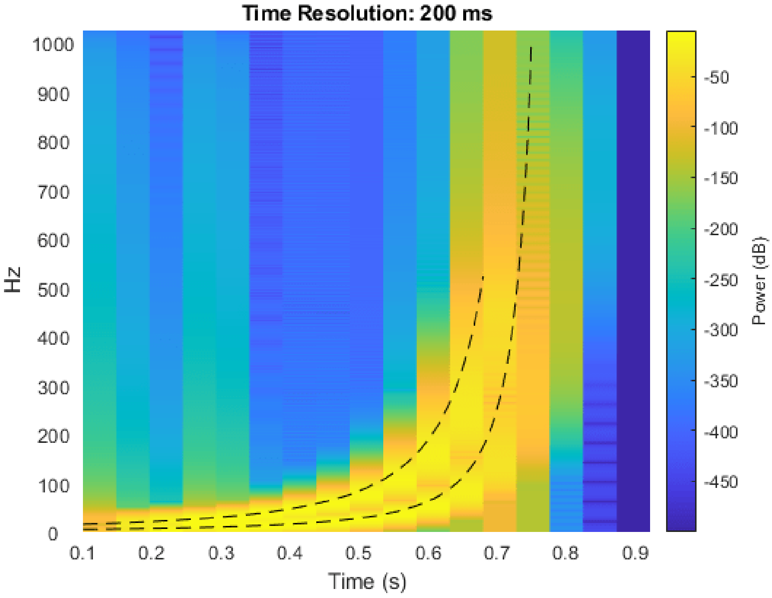
- sum of two hyperbolic chirps: one with instantaneous frequency $\frac{7.5}{(0.80-t)^2}$ and one with $\frac{2.5}{(0.80-t)^2}$ sampled at 2048Hz



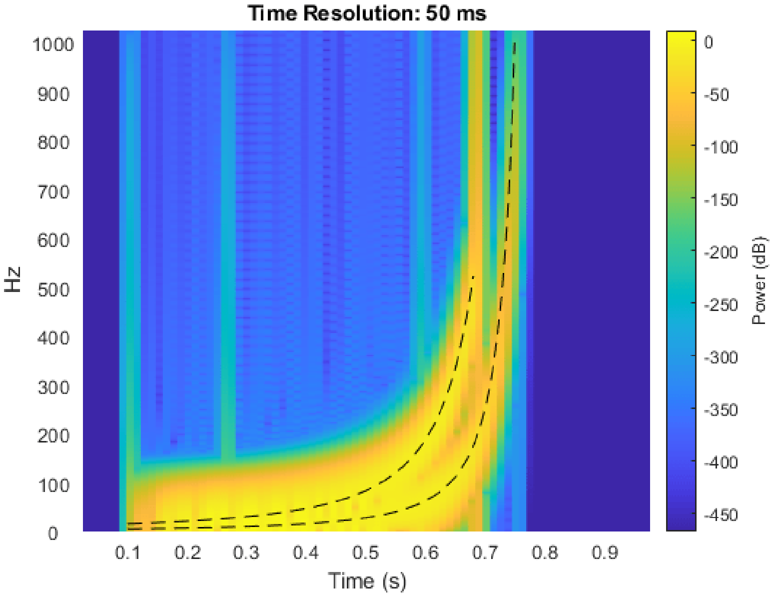
Discrete Fourier Transform



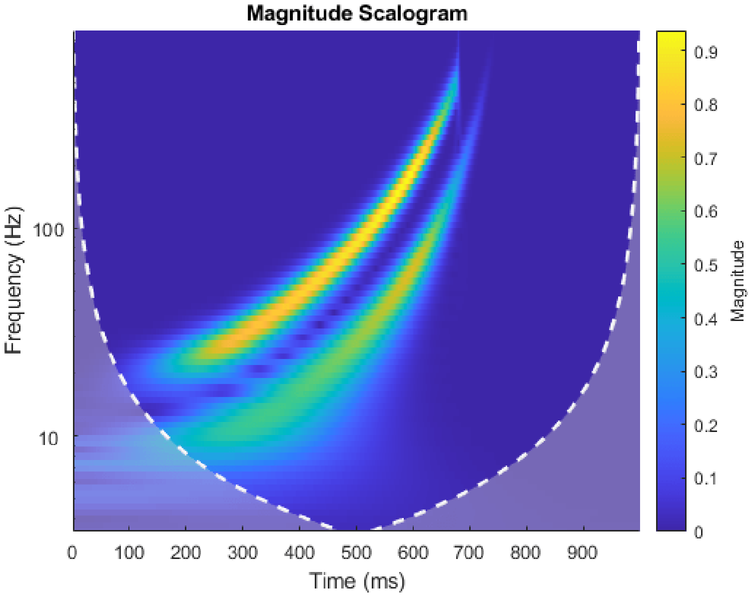
Short-Time Fourier Transform - long window



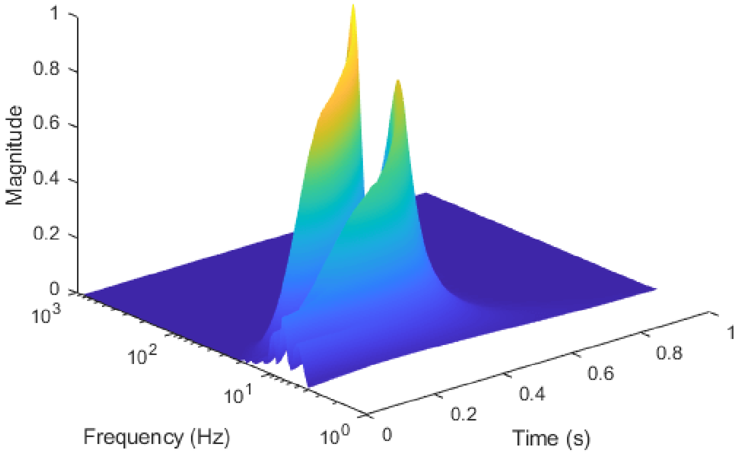
Short-Time Fourier Transform - short window



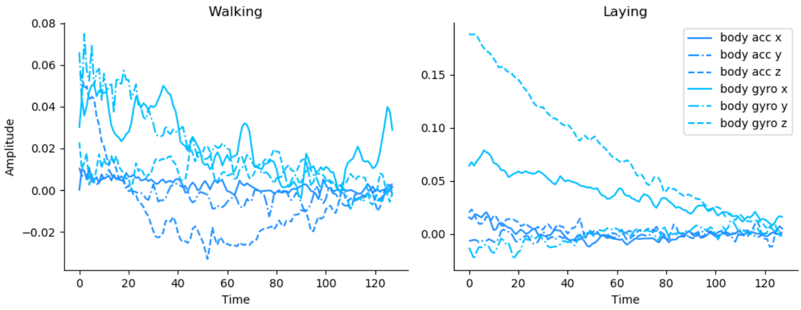
Continuous Wavelet Transform



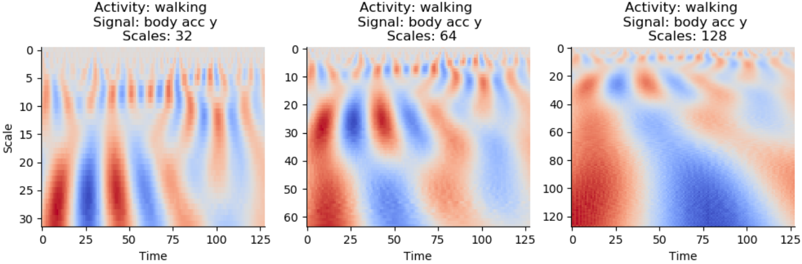
Scalogram In 3-D



Human Activity Recognition (HAR) Dataset

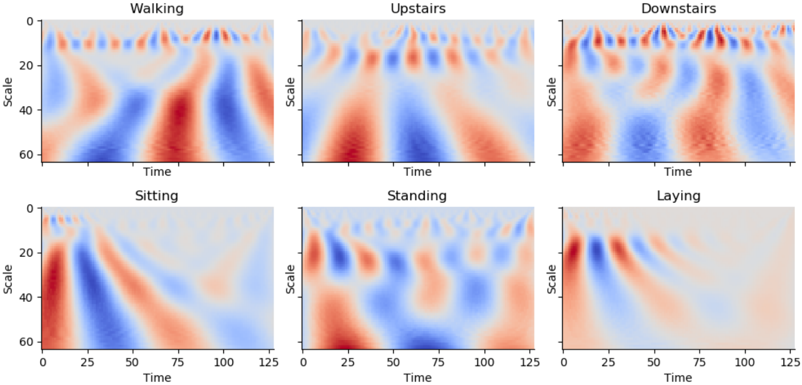


Continuous Wavelet Transform of HAR signals

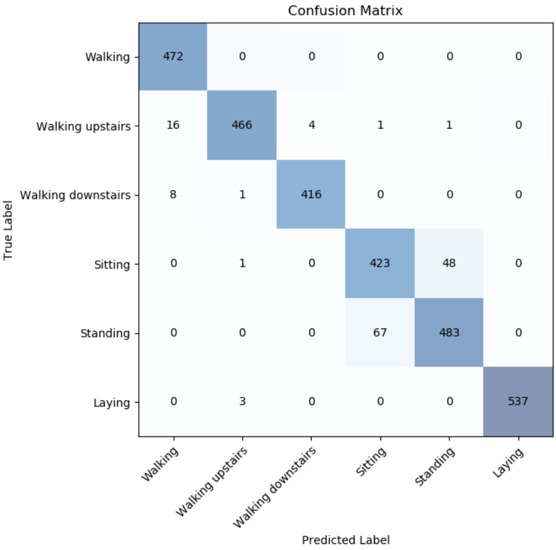


- changing the number of scales

Continuous Wavelet Transform of HAR signals



Deep Convolutional Neural Network

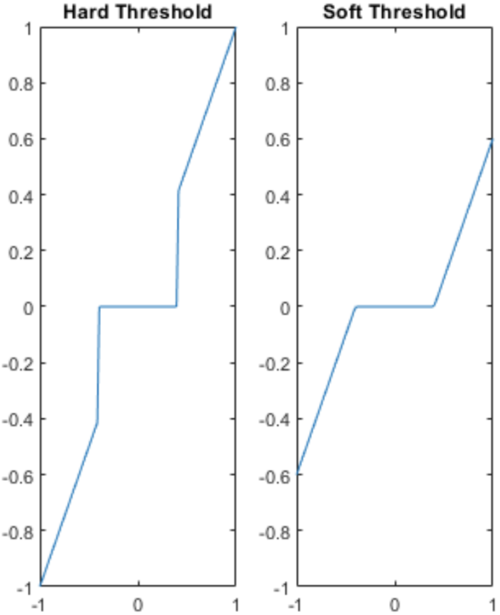


- 94.91% test accuracy

Wavelet Denoising

1. input signal $x = \left[x[1], \ldots, x[N]\right]$
2. compute discrete wavelet transform $y = Wx$
3. perform thresholding in the wavelet domain (shrink coefficients by hard/soft thresholding)
4. reconstruct the signal from thresholded discrete wavelet coefficients $\hat{x} = W^{-1}S(y)$

Hard/Soft Thresholding



Hard/Soft Thresholding

- hard thresholding

$$H_\lambda(y) = y1[|y| > \lambda]$$

- soft thresholding

$$S_\lambda(y) = \text{sign}(y)(|y| - \lambda)_+$$

Sparse Signal Recovery and Optimization

- noisy observation model

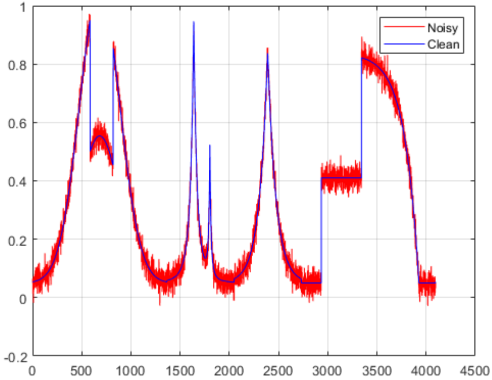
$$y = y^* + \sigma w$$

  $w \sim N(0,1)$ independent identically distributed Gaussian noise
- y^* is a sparse clean signal
- recover y^* from y via

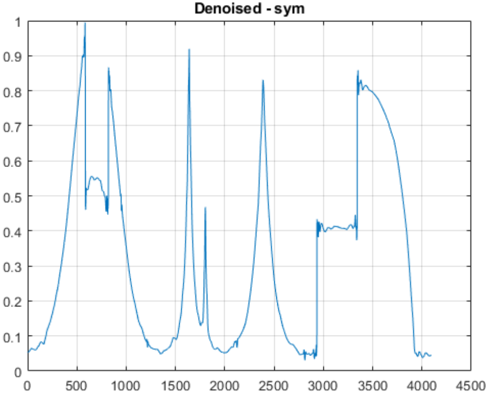
hard thresholding: $\hat{y} = H_\lambda(y) = \arg\min_z \|y - z\|_2^2 + \lambda\|z\|_0$

soft thresholding: $\hat{y} = S_\lambda(y) = \arg\min_z \|y - z\|_2^2 + \lambda\|z\|_1$

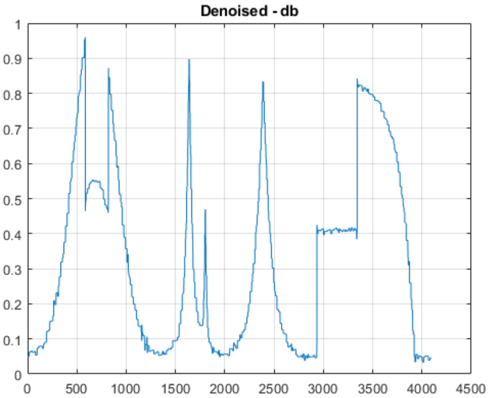
Wavelet Denoising



Wavelet Denoising



Wavelet Denoising



Hard vs Soft Thresholding

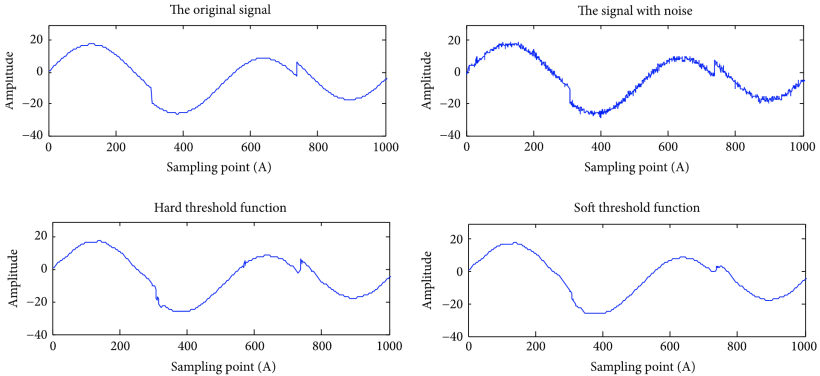
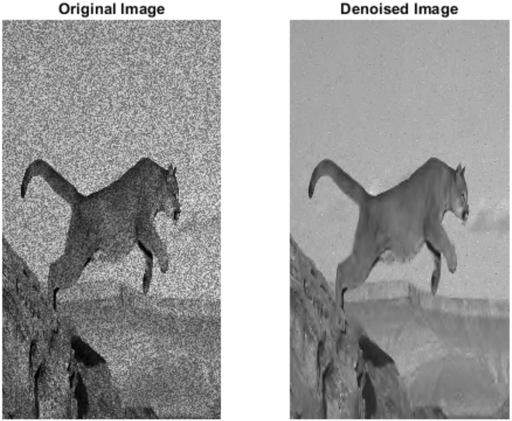


Image Denoising by Wavelet Thresholding



- 2D Biorthogonal mother wavelet