EE269
Signal Processing for Machine Learning

Wavelets, Discrete Wavelet Transform and Short-Time Fourier Transform

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Continuous Wavelet Transform

- Define a function $\psi(t)$
- Create scaled and shifted versions of $\psi(t)$

$$\psi_{s,\tau} = \frac{1}{\sqrt{s}}\psi(\frac{t-\tau}{s})$$

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$$W(s,\tau) = \int_{-\infty}^{\infty} f(t)\psi^*_{s,\tau}dt = \langle f(t), \psi_{s,\tau}\rangle$$

- Transforms a continuous function of one variable into a continuous function of two variables : **translation** and **scale**
- For a compact representation, we can choose a mother wavelet $\psi(t)$ that matches the signal shape

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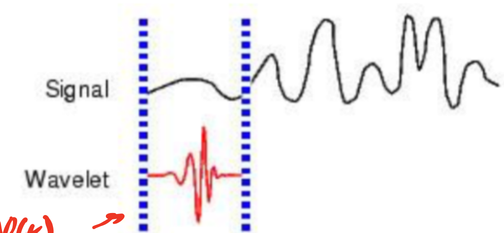
- Transforms a continuous function of one variable into a continuous function of two variables : **translation** and **scale**
- For a compact representation, we can choose a mother wavelet $\psi(t)$ that matches the signal shape
- Inverse Wavelet Transform

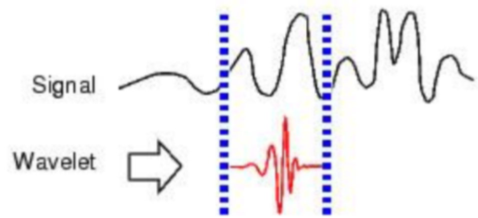
$$f(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(s,\tau)\psi_{s,\tau}d\tau ds$$

Wavelets

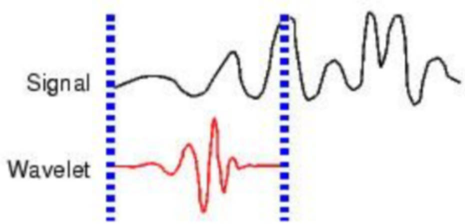


$\psi(t)$

Low value for $W_\psi(s,\tau)$



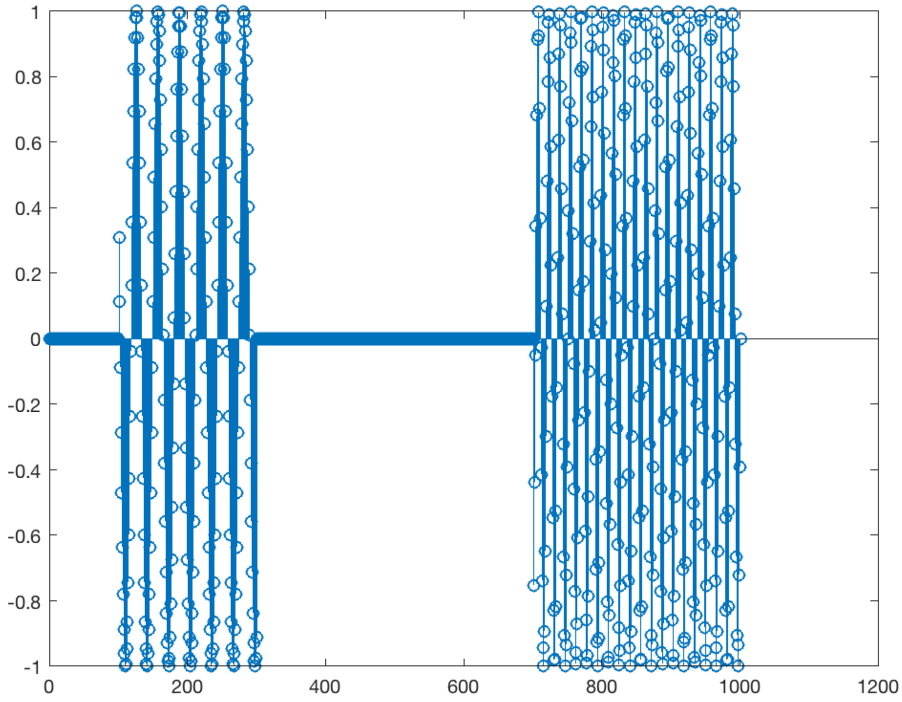
Higher value of $W_\psi(s,\tau_2)$



Different scale

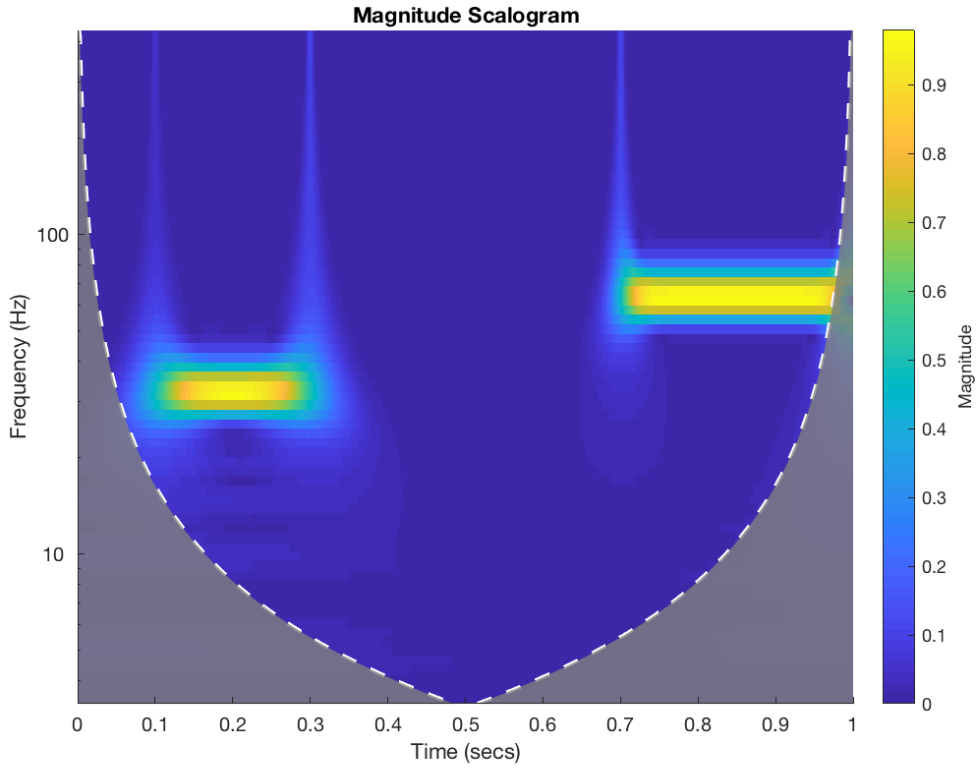
Continuous Wavelet Transform

```
Fs = 1e3;
t = 0:1/Fs:1;
x = cos(2*pi*32*t).*(t>=0.1 & t<0.3) + sin(2*pi*64*t).*(t>0.7);
stem(x);
```



Continuous Wavelet Transform

```
cwt(x,1000)
```

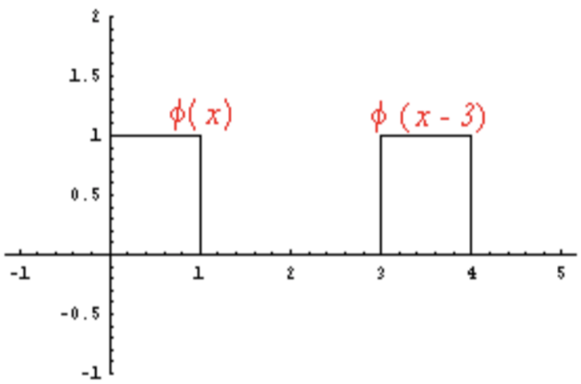


Continuous Haar Wavelets

- Consider the function

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- translates $\phi(x-k)$



Continuous Haar Wavelets

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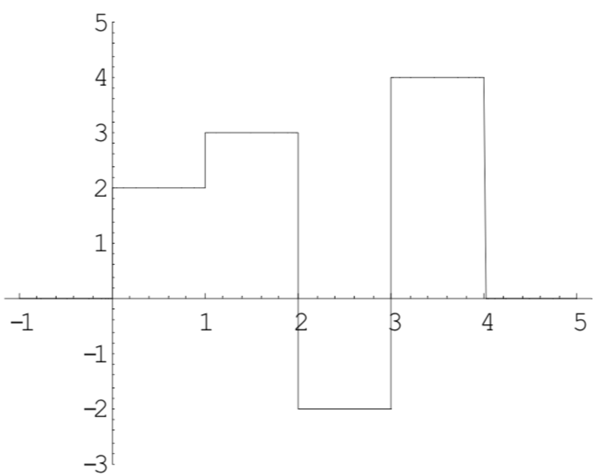
$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- linear combination of the translates $\phi(x-k)$



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- Consider the function

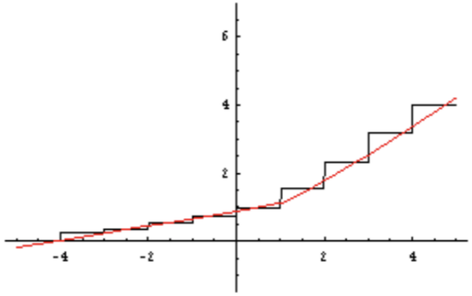
$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Define

$V_0 =$ all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(x - k)$$

$=$all square integrable functions which are constant on integer intervals



Continuous Haar Wavelets

- Consider the function

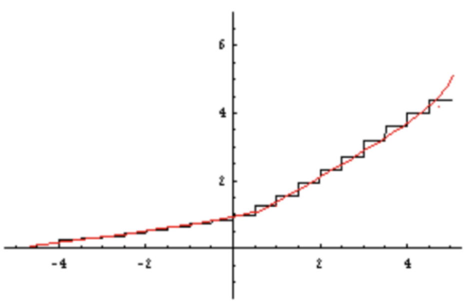
$$\phi(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- Define

$V_1 =$ all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(2x - k)$$

$=$all square integrable functions which are constant on half integer intervals



Continuous Haar Wavelets

- Consider the function

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- Define

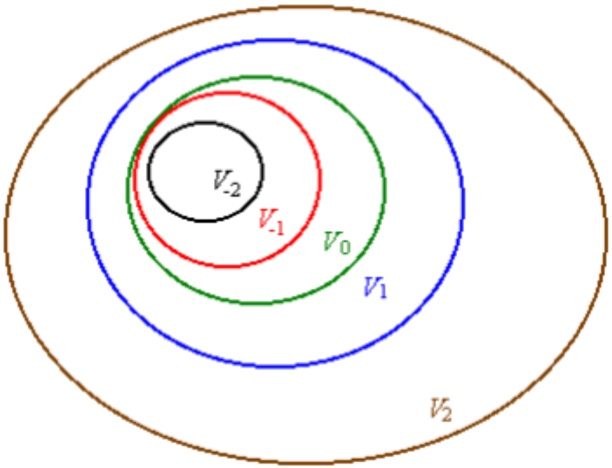
V_j = all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(2^j x - k)$$

=all square integrable functions which are constant on 2^{-j} length intervals

Continuous Haar Wavelets

- Nested spaces: $...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2...$



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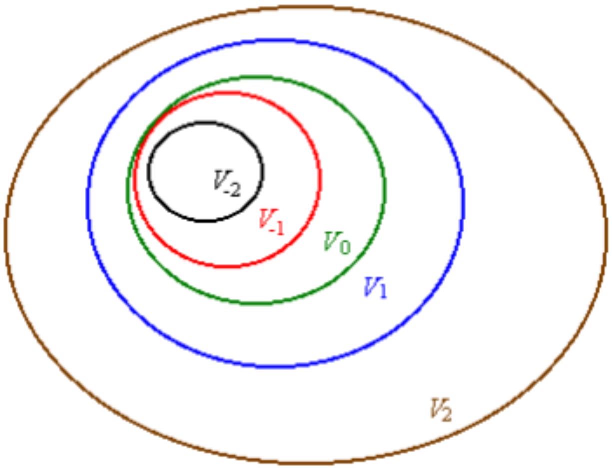

- There is a subspace W_0 such that $V_0 \oplus W_0 = V_1$, i.e. $W_0 := V_1 \ominus V_0$
- In general $W_j = V_{j+1} \ominus V_j$

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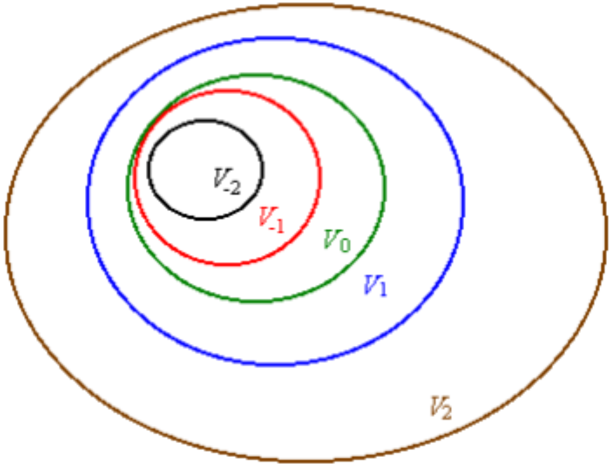

- $W_j = V_{j+1} \ominus V_j$
- **Theorem**: Every square integrable function can be uniquely expressed as

$$\sum_{j=-\infty}^{\infty} w_j \qquad \text{where } w_j \in W_j$$

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- Each function can be written as $f = \sum_j w_j$
- $f = \sum_j \sum_j a_{jk} \psi_{jk}(x)$ (multiresolution analysis)

Discrete Wavelet Transform

- Discrete shifts and scales $\psi(\frac{t-\tau}{s})$
- Suppose we have a signal of length N

$$x = [x_1, x_2, ...x_N]$$

- Consider a length $N/2$ approximation of x, e.g., for transmission

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- example

$$x = [6, 12, 15, 15, 14, 12, 120, 116] \rightarrow s = [9, 15, 13, 118]$$

- suppose that we are allowed to send $N/2$ more numbers
- differences

$$d_k = \frac{x_{2k-1} - x_{2k}}{2}, \quad k = 1, ..., N/2$$

- we can recover x

$$x = [6, 12, 15, 15, 14, 12, 120, 116] \rightarrow$$
$$[s \mid d] = [9, 15, 13, 118 \mid 3, 0, -1, -2]$$

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$$[s \mid d] = [9, 15, 13, 118 \mid 3, 0, -1, -2]$$

- One step Haar Transformation $x \rightarrow [s|d]$

One Step Haar Transformation

$$\left[\begin{array}{cccccccc}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
\hline
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2}
\end{array}\right] \cdot \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{array}\right] = \frac{1}{2}\left[\begin{array}{c} x_1 + x_2 \\ x_3 + x_4 \\ x_5 + x_6 \\ x_7 + x_8 \\ \hline x_2 - x_1 \\ x_4 - x_3 \\ x_6 - x_5 \\ x_8 - x_7 \end{array}\right]$$

Discrete Haar Wavelet Transform

56	40	8	24	48	48	40	16

Discrete Haar Wavelet Transform

$$\frac{56+40}{2}$$

$$56-48$$

56	40	8	24	48	48	40	16
48				8			

Discrete Haar Wavelet Transform

56	40	8	24	48	48	40	16
48	16			8	-8		

Discrete Haar Wavelet Transform

56	40	8	24	48	48	40	16
48	16	48		8	-8	0	

Discrete Haar Wavelet Transform

56	40	8	24	48	48	40	16
48	16	48	28	8	-8	0	12

Discrete Haar Wavelet Transform

56	40	8	24	48	48	40	16
48	16	48	28	8	-8	0	12
				8	-8	0	12

Repeating the same process on the averages

56	40	8	24	48	48	40	16
48	16	48	28	8	−8	0	12
32		16		8	−8	0	12

Discrete Haar Wavelet Transform

56	40	8	24	48	48	40	16
48	16	48	28	8	-8	0	12
32	38	16	10	8	-8	0	12

Discrete Haar Wavelet Transform

56	40	8	24	48	48	40	16
48	16	48	28	8	−8	0	12
32	38	16	10	8	−8	0	12
		16	10	8	−8	0	12

Discrete Haar Wavelet Transform

56	40	8	24	48	48	40	16
48	16	48	28	8	−8	0	12
32	38	16	10	8	−8	0	12
35	−3	16	10	8	−8	0	12

Discrete Haar Wavelet Transform

Discrete Wavelet Transform : Haar wavelet

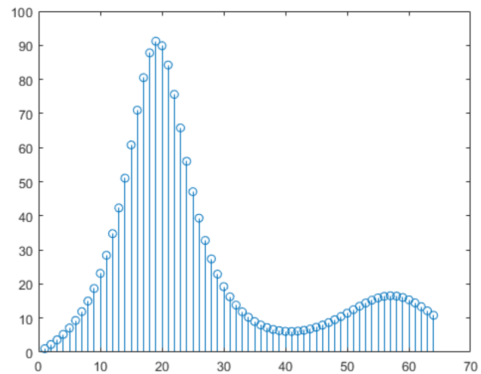
```
s = 1;
h = [1  1]/2;
g = [1 -1]/2;

y = [2 4 6 8]
a1 = filter (h, 1, y)                % level 1 approximation
d1 = filter (g, 1, y)                % level 1 detail

a1 = a1 (2:2:end)                    % downsample
d1 = d1 (2:2:end)

a2 = filter (h, 1, a1)               % level 2 approximation
d2 = filter (g, 1, a1)               % level 2 detail
a2 = a2 (2:2:end)
d2 = d2 (2:2:end)                    % downsample


dwty = [a2 d2 d1]
```

```
y = 1×4
      2     4     6     8

a1 = 1×4
      1     3     5     7

d1 = 1×4
      1     1     1     1

a1 = 1×2
      3     7

d1 = 1×2
      1     1

a2 = 1×2
    1.5000    5.0000

d2 = 1×2
    1.5000    2.0000

a2 = 5
d2 = 2
dwty = 1×4
      5     2     1     1
```

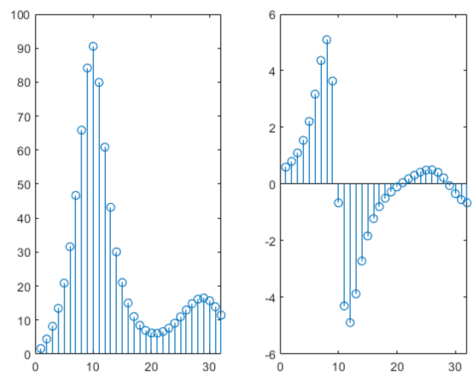
Discrete Haar Wavelet Transform

let's try another signal

```
x = (1:64) / 64;
y = humps(x) - humps(0);
stem(y)
```

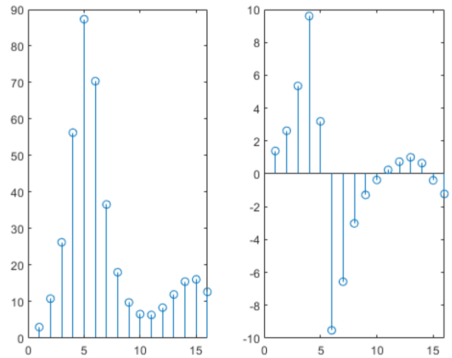



```
a1 = filter (h, 1, y);              % level 1 approximation
d1 = filter (g, 1, y);              % level 1 detail

a1 = a1 (2:2:end);                  % downsample
d1 = d1 (2:2:end);
subplot(1,2,1);stem(a1);subplot(1,2,2);stem(d1)
```

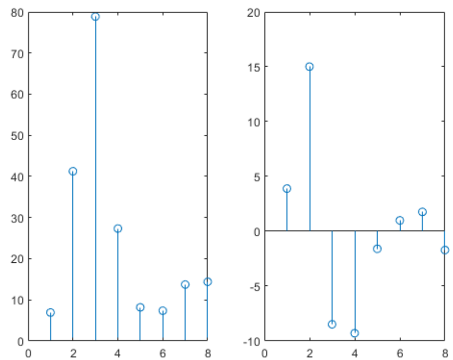


Discrete Haar Wavelet Transform

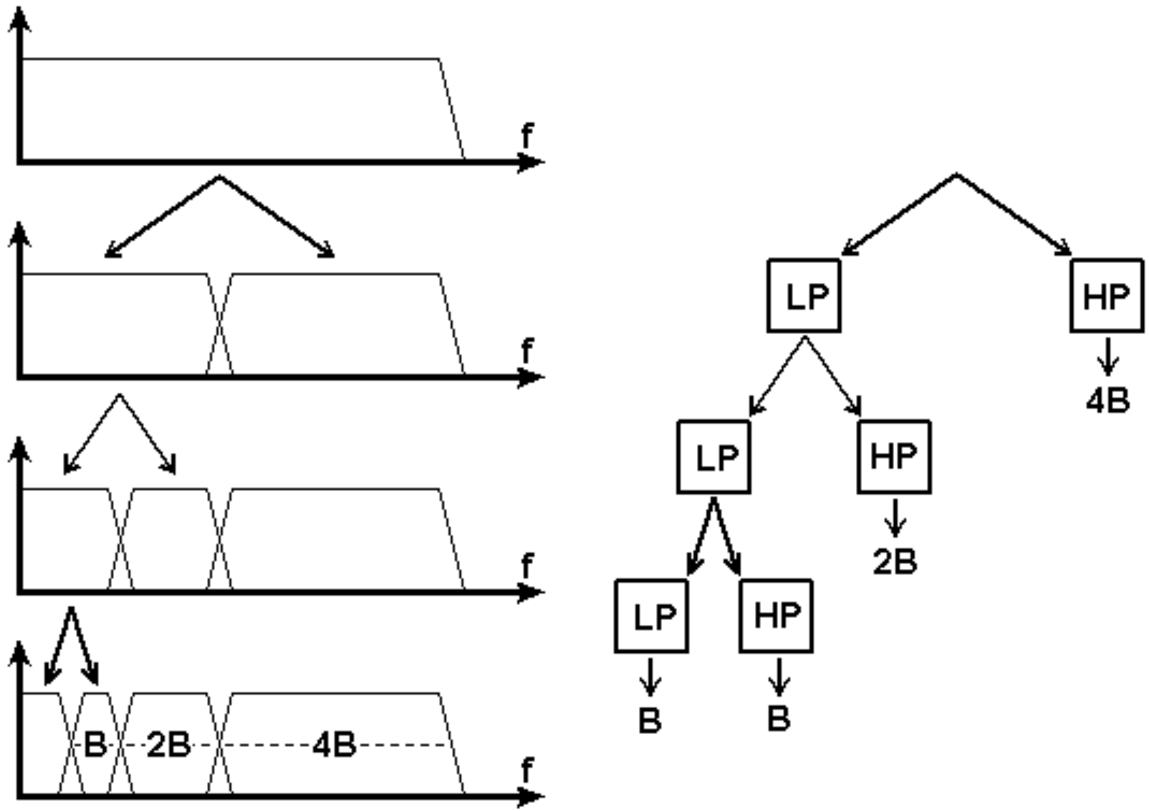
```
a2 = filter (h, 1, a1);              % level 2 approximation
d2 = filter (g, 1, a1);              % level 2 detail
a2 = a2 (2:2:end);
d2 = d2 (2:2:end);                   % downsample
figure;subplot(1,2,1);stem(a2);subplot(1,2,2);stem(d2)
```



```
a3 = filter (h, 1, a2);              % level 3 approximation
d3 = filter (g, 1, a2);              % level 3 detail
a3 = a3 (2:2:end);
d3 = d3 (2:2:end);                  % downsample
figure;subplot(1,2,1);stem(a3);subplot(1,2,2);stem(d3)
```



Discrete Haar Wavelet Transform



Discrete Haar Transform Filter Bank

$$H = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Discrete Haar Transform Matrix

- repeat the computation on the **means**
- keep differences in each step

2D Discrete Haar Transform

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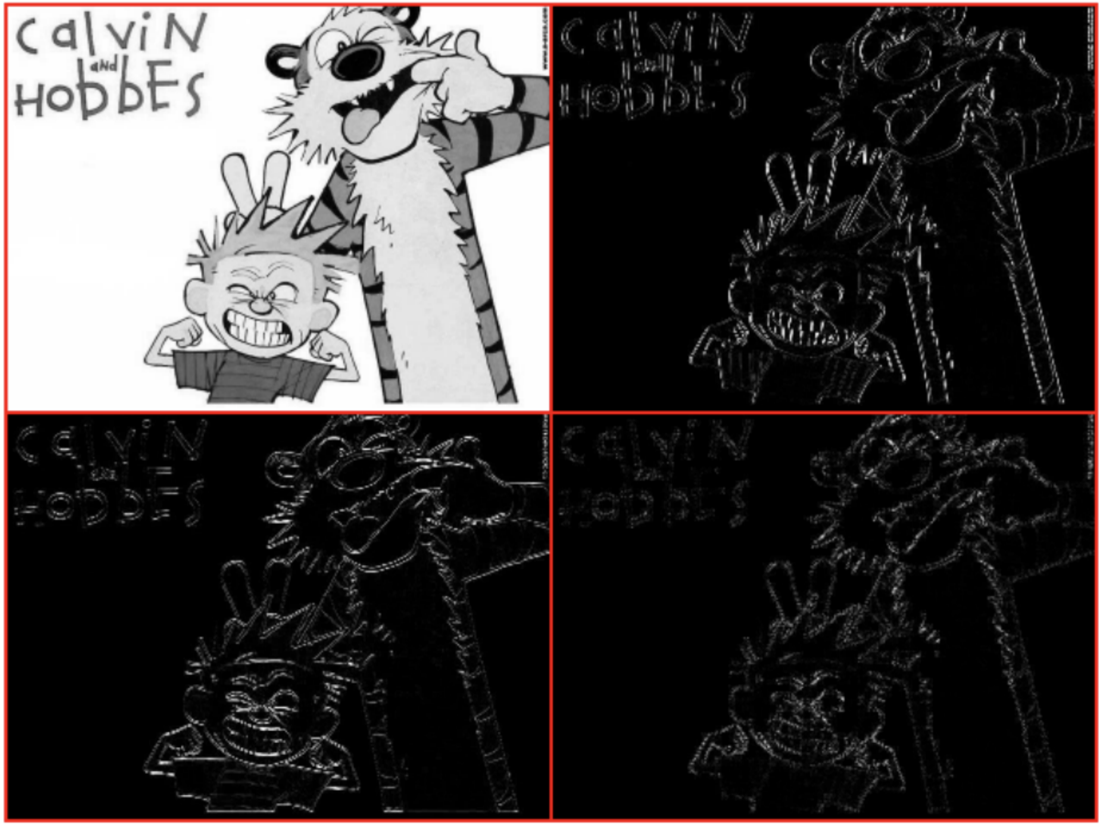

2D Discrete Haar Transform