EE 276: Information Theory

Polar Codes

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Outline

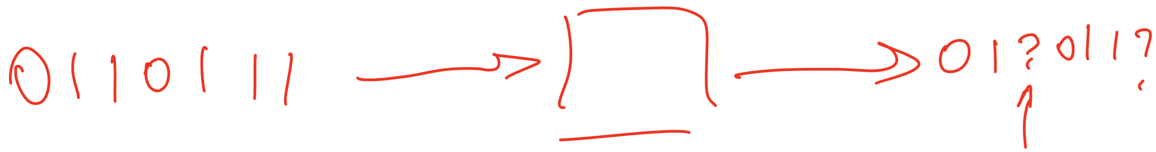
- Polar code construction
- Achieving channel capacity
- Decoding
- Applications and Extensions

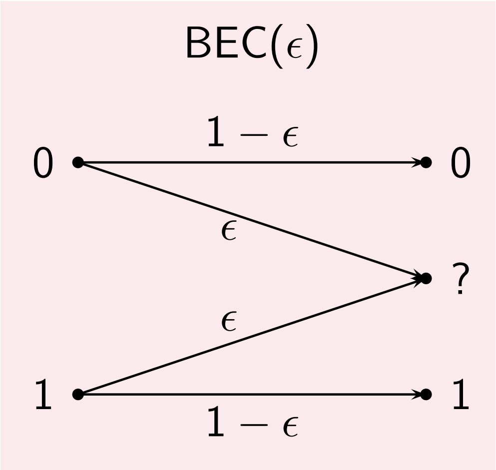
Channel Capacity

- Channel capacity C is the maximal rate of reliable communication
- Shannon's Second Fundamental Theorem (from Lecture 7) :

$$C = \max_{P_X} I(X;Y)$$

Capacity of the binary erasure channel (BEC)





Capacity of the BEC with erasure probability ϵ is $C = 1 - \epsilon$

Channel Coding

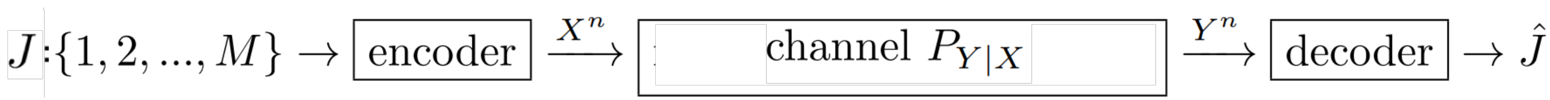
$J : \{1, 2, ..., M\} \rightarrow$ [encoder] $\xrightarrow{X^n}$ [channel $P_{Y|X}$] $\xrightarrow{Y^n}$ [decoder] $\rightarrow \hat{J}$

$K = \log_2 M$

rate: $R = \frac{\log M}{n}$ bits/channel use

probability of error $P_{\text{error}} =$ Probability$[\hat{J} \neq J]$

Channel Coding



rate: $R = \frac{\log M}{n}$ bits/channel use

probability of error $P_{\text{error}} = \text{Probability}[\hat{J} \neq J]$

- If $R < C$, then there exists a communication scheme with rate $\geq R$ and probability of error: $P_{\text{error}} \to 0$

Channel Coding

$$J:\{1,2,...,M\} \rightarrow \boxed{\text{encoder}} \xrightarrow{X^n} \boxed{\text{channel } P_{Y|X}} \xrightarrow{Y^n} \boxed{\text{decoder}} \rightarrow \hat{J}$$

rate: $R = \frac{\log M}{n}$ bits/channel use

probability of error $P_{\text{error}} =$ Probability$[\hat{J} \neq J]$

- If $R < C$, then there exists a communication scheme with rate $\geq R$ and probability of error: $P_{\text{error}} \rightarrow 0$
- If $R > C$, then rate R is not achievable (P_{error} is large)

Shannon's Second Theorem: Maximum rate of reliable communication is $C = \max_{P_X} I(X;Y)$

Shannon's Coding Method

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0 1 1 0

0 1 ? ?

- random codebook (from Lecture 11) 0 1 1 0 1

codeword 1	0	1	1	0	1	...
codeword 2	1	1	0	1	0	...
codeword 3	1	1	0	1	0	...
codeword 4	0	0	0	0	1	...
codeword 5	1	0	0	0	0	...
codeword 6	0	1	0	0	1	...
codeword 7	0	0	1	1	0	...
codeword 8	1	0	0	0	1	...
⋮						

- not explicitly constructed
- shows the existence of good codes
- not computationally efficient

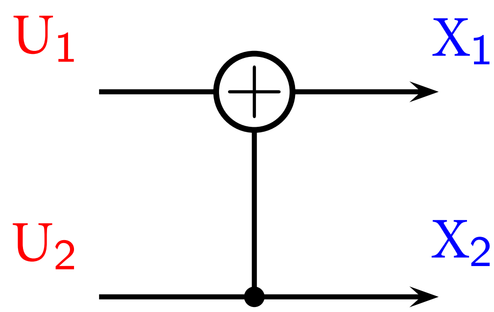
"Almost all codes are "good" codes except for the the ones that we can think of..."
Jack Wolfe

Today: Polar Codes

- Invented by Erdal Arıkan in 2009
- First code with an explicit construction to provably achieve the channel capacity
- Efficient encoding/decoding operations

(channel coding vs source coding)

Basic 2×2 transformation



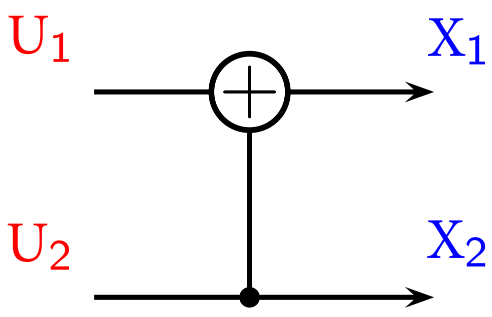
$X_1 = U_1 \oplus U_2 = U_1 + U_2 \mod 2$

XOR

$U_1 \oplus U_2$	$U_2 = 0$	$U_2 = 1$
$U_1 = 0$	0	1
$U_1 = 1$	1	0

$U_1, U_2 \in \{0, 1\}$ two input bits
$X_1, X_2 \in \{0, 1\}$ two output bits

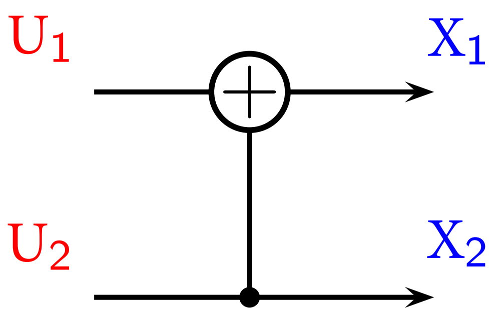
Basic 2×2 transformation



$U_1, U_2 \in \{0,1\}$ two input bits
$X_1, X_2 \in \{0,1\}$ two output bits

$$X_1 = U_1 \oplus U_2 = \; U_1 \text{ XOR } U_2$$
$$X_2 = U_2$$

Basic 2×2 transformation



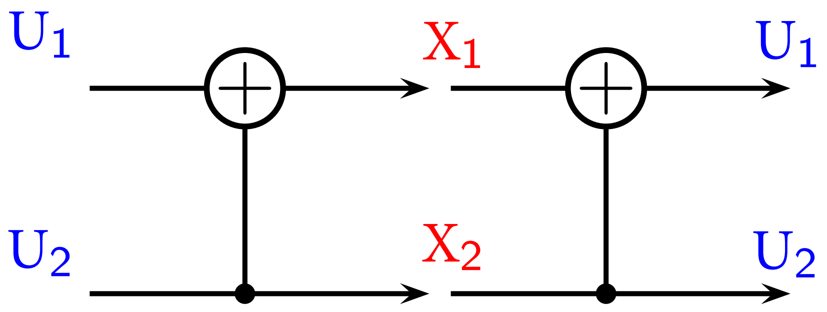
$U_1, U_2 \in \{0,1\}$ two input bits
$X_1, X_2 \in \{0,1\}$ two output bits

$$X_1 = U_1 \oplus U_2 = \; U_1 \text{ XOR } U_2$$
$$X_2 = U_2$$

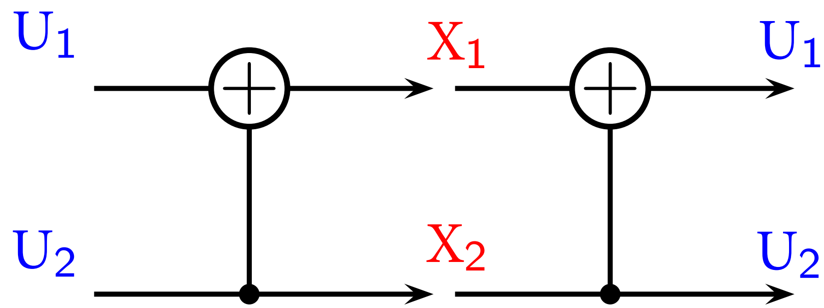
alternatively

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad \text{modulo } 2$$

Inverting the transform



Inverting the transform

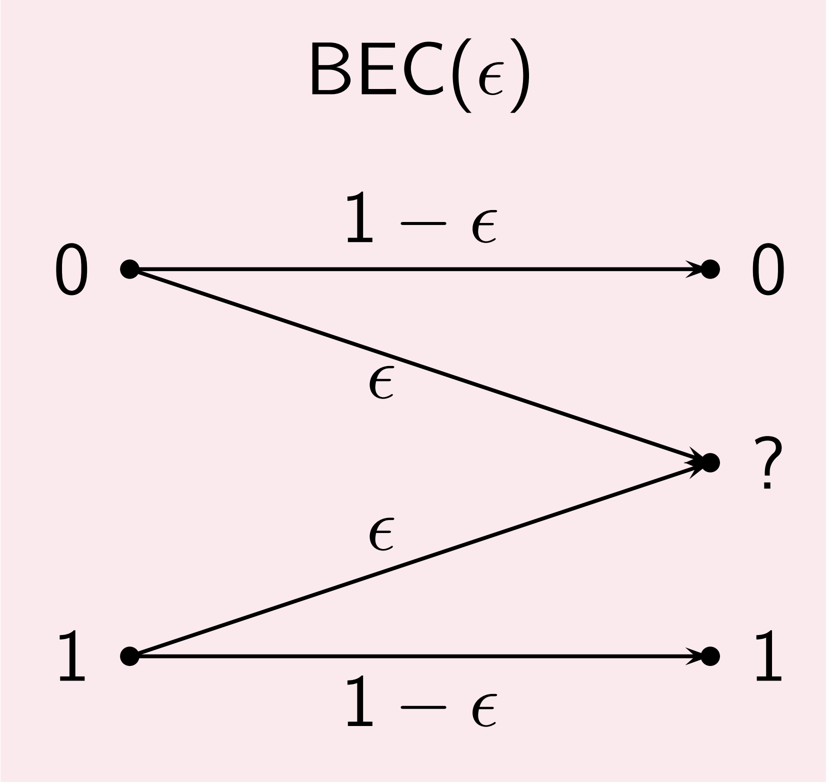


$$2 \times 2 \text{ transformation} \qquad G_2 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$G_2 G_2 U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \oplus U_2 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_1 \oplus U_2 \oplus U_2 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

Erasure channel



Naively combining erasure channels

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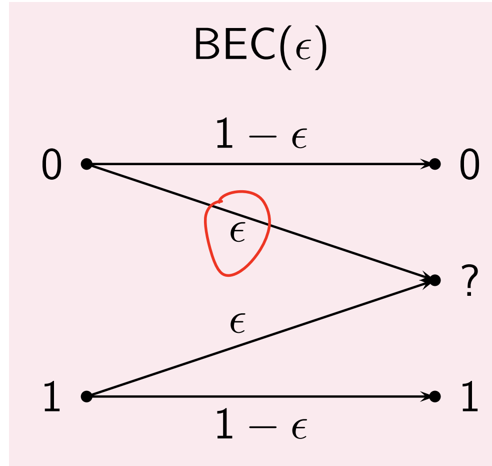
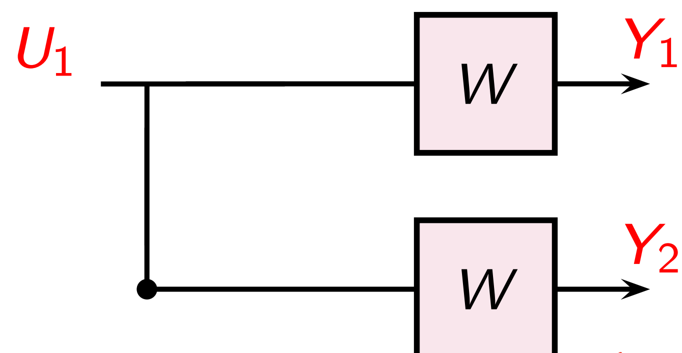


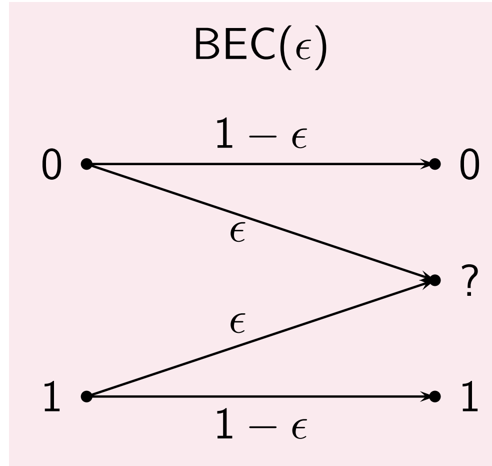
- Repetition coding



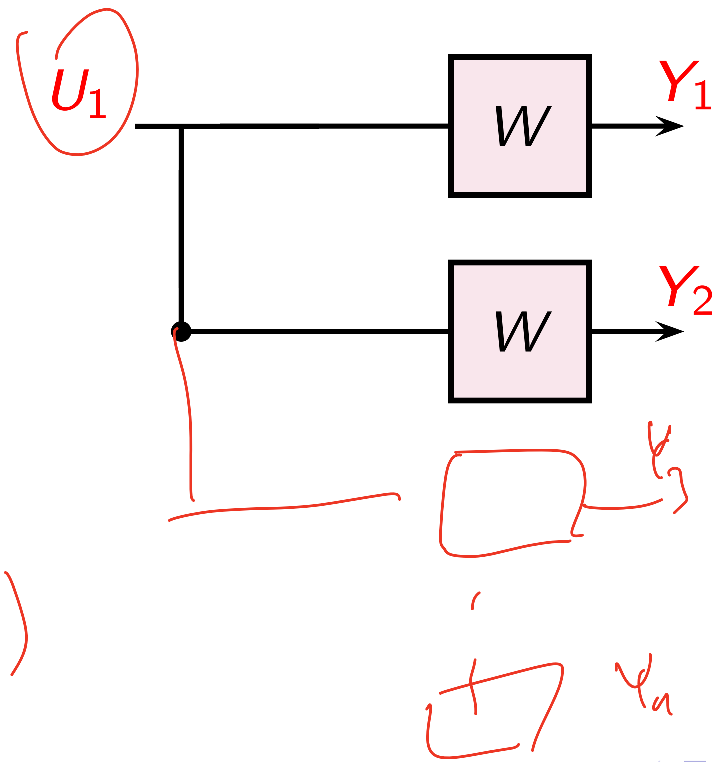
$P(U_1 \text{ is erased}) = P(Y_1 \text{ erased } \& \; Y_2 \text{ erased})$

$= \epsilon \cdot \epsilon = \epsilon^2$

Repetition code with rate $1/n$



- Repetition coding



0.1

$(0.1)^n$

$\text{Rate} = \frac{1}{n}$

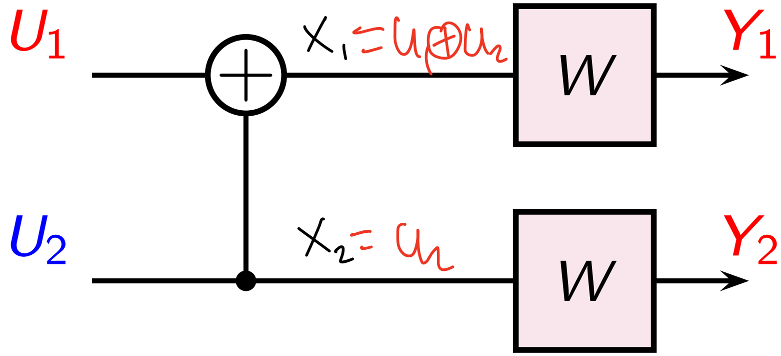
$\text{Rate} = 0.5$

$P(U_1 \text{ is erased}) =$

$P(\text{all } Y_i \text{ erased } i \in [n])$

$= \epsilon^n$

Combining two erasure channels



Invertible transformation does not alter capacity or mutual information: $I(U;Y) = I(X;Y)$

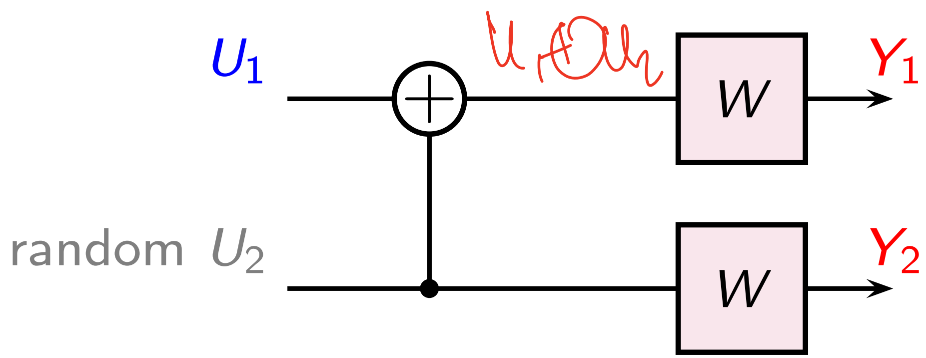
Sequential decoding: Decode U_1 and U_2 one by one

$U_1 \oplus U_2 \oplus U_2 = U_1$

$2 \cdot 0.1 - (0.1)^2$

First bit-channel $W_1 : U_1 \to (Y_1, Y_2)$

$\epsilon = 0.1$



$P(U_1 \text{ is erased}) = P(Y_1 \text{ erased or } Y_2 \text{ erased})$

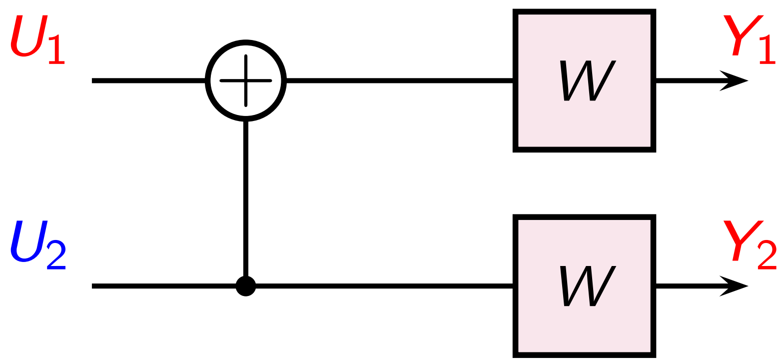
$= 1 - (1-\epsilon)^2$

$= 1 - (1 - 2\epsilon + \epsilon^2) = 2\epsilon - \epsilon^2$

probability	Y_1 erased	Y_1 not erased
Y_2 erased	$\epsilon \cdot \epsilon$	$(1-\epsilon)\epsilon$
Y_2 not erased	$(1-\epsilon)\epsilon$	$(1-\epsilon)^2$

suppose a '**genie**' provides this bit

Second bit-channel $W_2 : U_2 \to (Y_1, Y_2, U_1)$

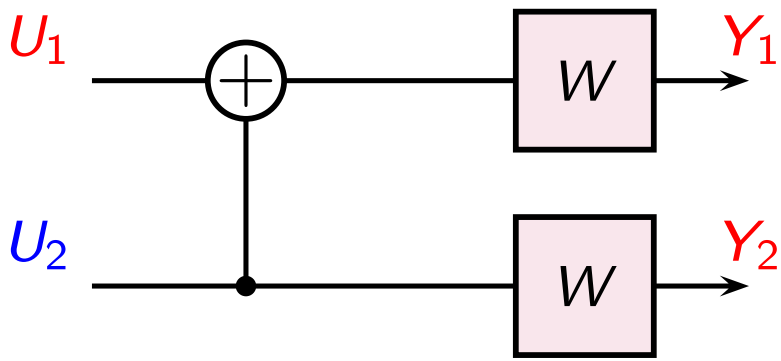


suppose a **'genie'** provides this bit

Second bit-channel $W_2 : U_2 \to (Y_1, Y_2, U_1)$

GPT4 prompt: generate an artistic depiction of a genie aided channel decoder

Second bit-channel $W_2 : U_2 \to (Y_1, Y_2, U_1)$

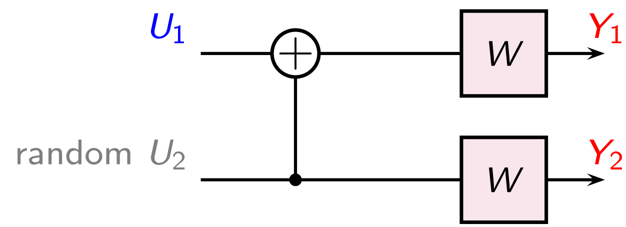


$P(U_2 \text{ is erased}) = \epsilon^2$

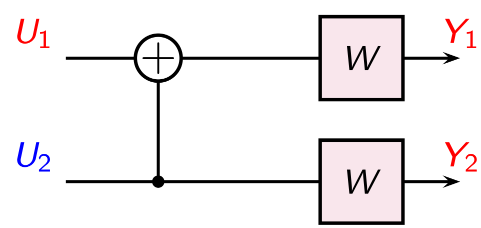
probability	Y_1 erased	Y_1 not erased
Y_2 erased	ϵ^2	$\epsilon(1-\epsilon)$
Y_2 not erased	$\epsilon(1-\epsilon)$	$(1-\epsilon)^2$

Two different cases: W_1 and W_2

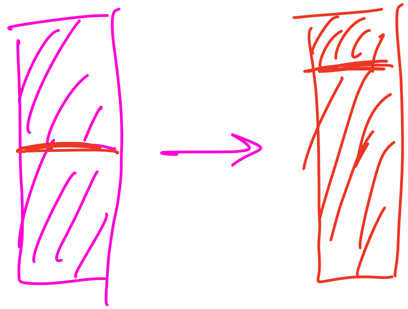
- W_1: Decoding U_1 when U_2 is **not** available
  U_1 is erased when Y_1 is erased **or** Y_2 is erased
  Failure probability $= 1 - (1-\epsilon)^2 = 2\epsilon - \epsilon^2$ **worse erasure channel**



- W_2: Decoding U_2 when U_1 is **available**
  U_2 is erased when Y_1 is erased **and** Y_2 is erased
  Failure probability $= \epsilon^2$ **better erasure channel**

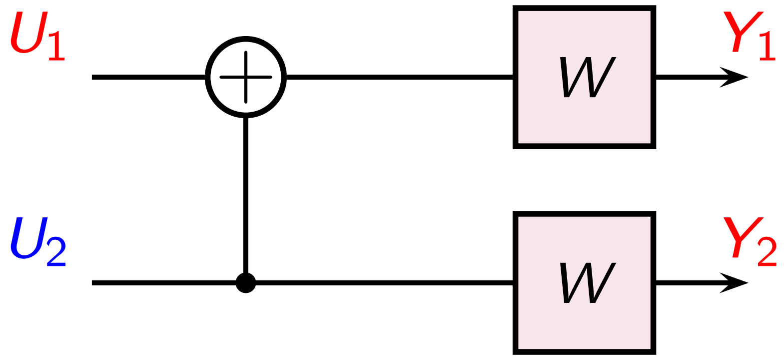


Capacity is conserved

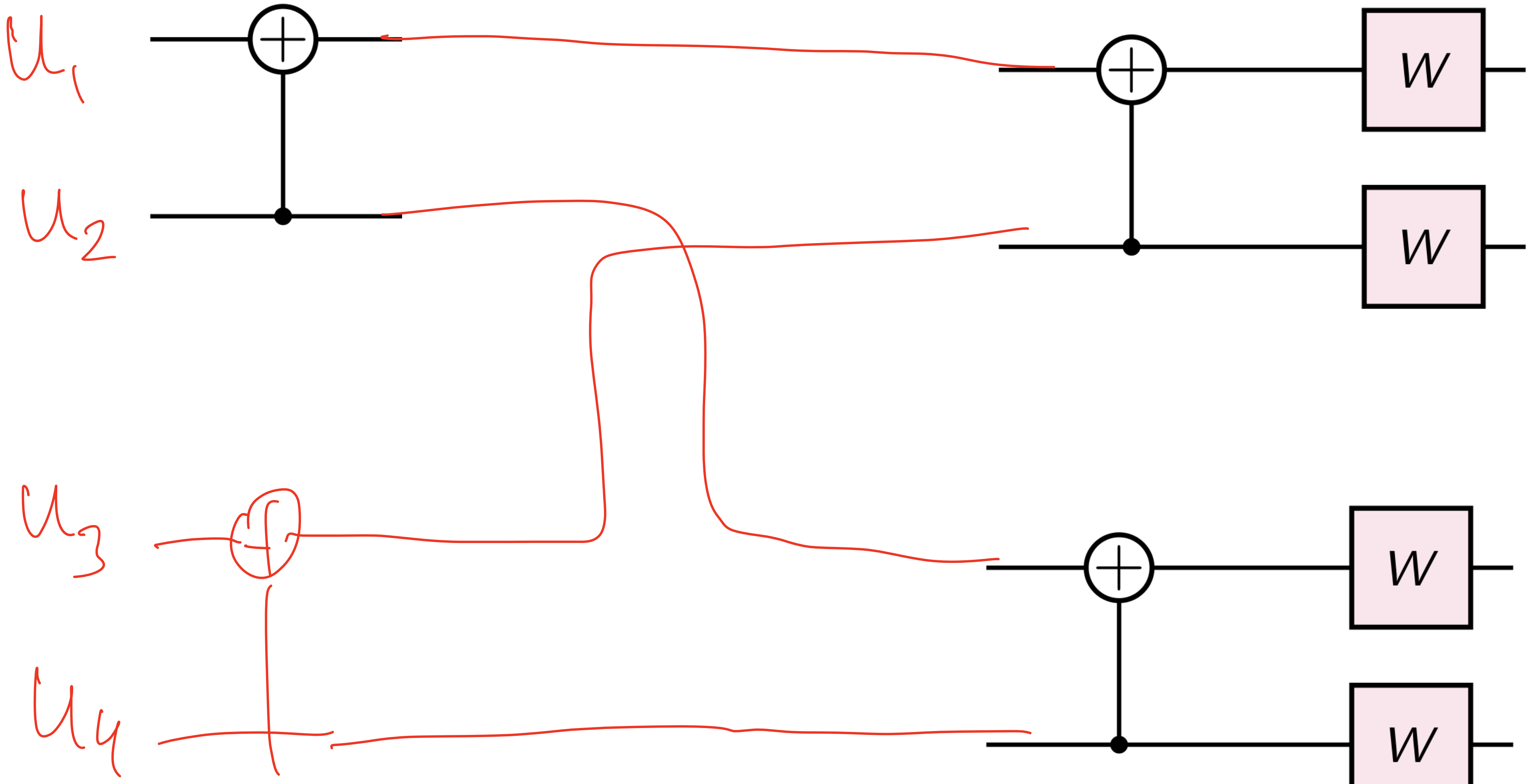


$$C(W_1) + C(W_2) = C(W) + C(W) = 2C(W)$$

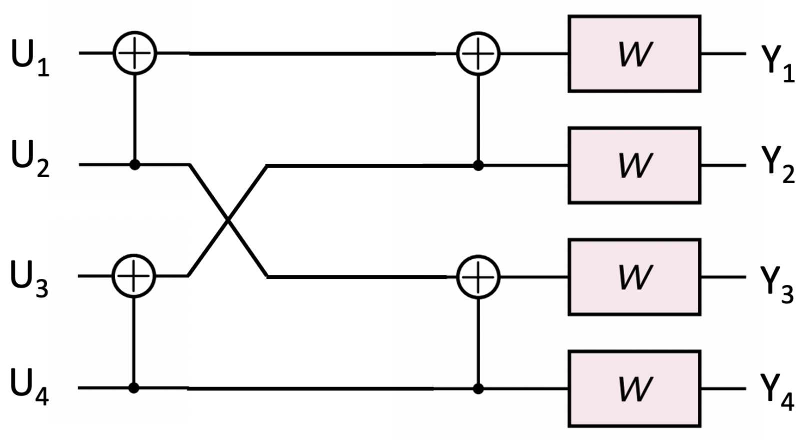
$$C(W_1) \leq C(W) \leq C(W_2)$$



Extending the size



Extending the size

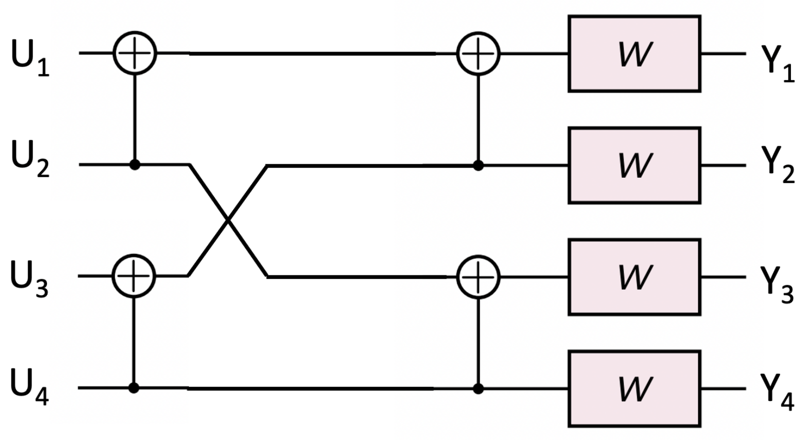


Sequential decoding:

- Decode U_1 from Y_1, Y_2, Y_3, Y_4

  erased if (Y_1 or Y_2 erased) **or** (Y_3 or Y_4 erased)

Extending the size



Sequential decoding:

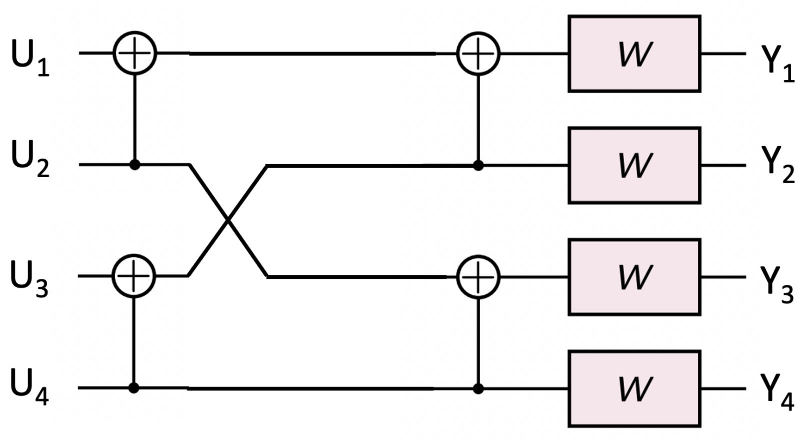
- Decode U_1 from Y_1, Y_2, Y_3, Y_4

  erased if (Y_1 or Y_2 erased) **or** (Y_3 or Y_4 erased)

- Decode U_2 from Y_1, Y_2, Y_3, Y_4, U_1

  erased if (Y_1 or Y_2 is erased) **and** (Y_3 or Y_4 is erased)

Extending the size



Sequential decoding:

- Decode U_1 from Y_1, Y_2, Y_3, Y_4

  erased if (Y_1 or Y_2 erased) **or** (Y_3 or Y_4 erased)
- Decode U_2 from Y_1, Y_2, Y_3, Y_4, U_1

  erased if (Y_1 or Y_2 is erased) **and** (Y_3 or Y_4 is erased)
- Decode U_3 from $Y_1, Y_2, Y_3, Y_4, U_1, U_2$

  erased if (Y_1 and Y_2 is erased) **or** (Y_3 and Y_4 is erased)
- Decode U_4 from $Y_1, Y_2, Y_3, Y_4, U_1, U_2, U_3$

  erased if (Y_1 and Y_2 erased) **and** (Y_3 and Y_4 erased)

Recursive Calculation of Failure Probability

Sequential decoding:

$\hat{\epsilon} = 2\epsilon - \epsilon^2$

- Decode U_1

  erased if (Y_1 or Y_2 erased) **or** (Y_3 or Y_4 erased)

failure probability$= 2\hat{\epsilon} - \hat{\epsilon}^2 = 2(2\epsilon - \epsilon^2) - (2\epsilon - \epsilon^2)^2$

Recursive Calculation of Failure Probability

Sequential decoding:

- Decode U_1

  erased if (Y_1 or Y_2 erased) **or** (Y_3 or Y_4 erased)

  failure probability$= 2\hat{\epsilon} - \hat{\epsilon}^2 = 2(2\epsilon - \epsilon^2) - (2\epsilon - \epsilon^2)^2$

- Decode U_2

  erased if (Y_1 or Y_2 is erased) **and** (Y_3 or Y_4 is erased)

  failure probability$= \hat{\epsilon}^2 = (2\epsilon - \epsilon^2)^2$

Recursive Calculation of Failure Probability

Sequential decoding:

- Decode U_1

  erased if (Y_1 or Y_2 erased) **or** (Y_3 or Y_4 erased)

  failure probability$= 2\hat{\epsilon} - \hat{\epsilon}^2 = 2(2\epsilon - \epsilon^2) - (2\epsilon - \epsilon^2)^2$
- Decode U_2

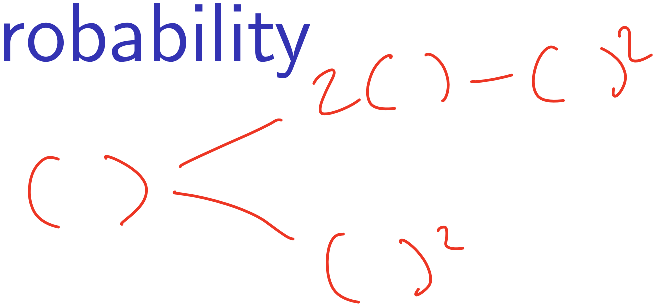
  erased if (Y_1 or Y_2 is erased) **and** (Y_3 or Y_4 is erased)

  failure probability$= \hat{\epsilon}^2 = (2\epsilon - \epsilon^2)^2$
- Decode U_3

  erased if (Y_1 and Y_2 is erased) **or** (Y_3 and Y_4 is erased)

  failure probability$= 2\tilde{\epsilon} - \tilde{\epsilon}^2 = 2(\epsilon^2) - (\epsilon^2)^2$

Recursive Calculation of Failure Probability



Sequential decoding:

- Decode U_1

  erased if (Y_1 or Y_2 erased) **or** (Y_3 or Y_4 erased)

  failure probability$= 2\hat{\epsilon} - \hat{\epsilon}^2 = 2(2\epsilon - \epsilon^2) - (2\epsilon - \epsilon^2)^2$

- Decode U_2

  erased if (Y_1 or Y_2 is erased) **and** (Y_3 or Y_4 is erased)

  failure probability$= \hat{\epsilon}^2 = (2\epsilon - \epsilon^2)^2$

- Decode U_3

  $\tilde{\epsilon} = \epsilon^2$

  erased if (Y_1 and Y_2 is erased) **or** (Y_3 and Y_4 is erased)

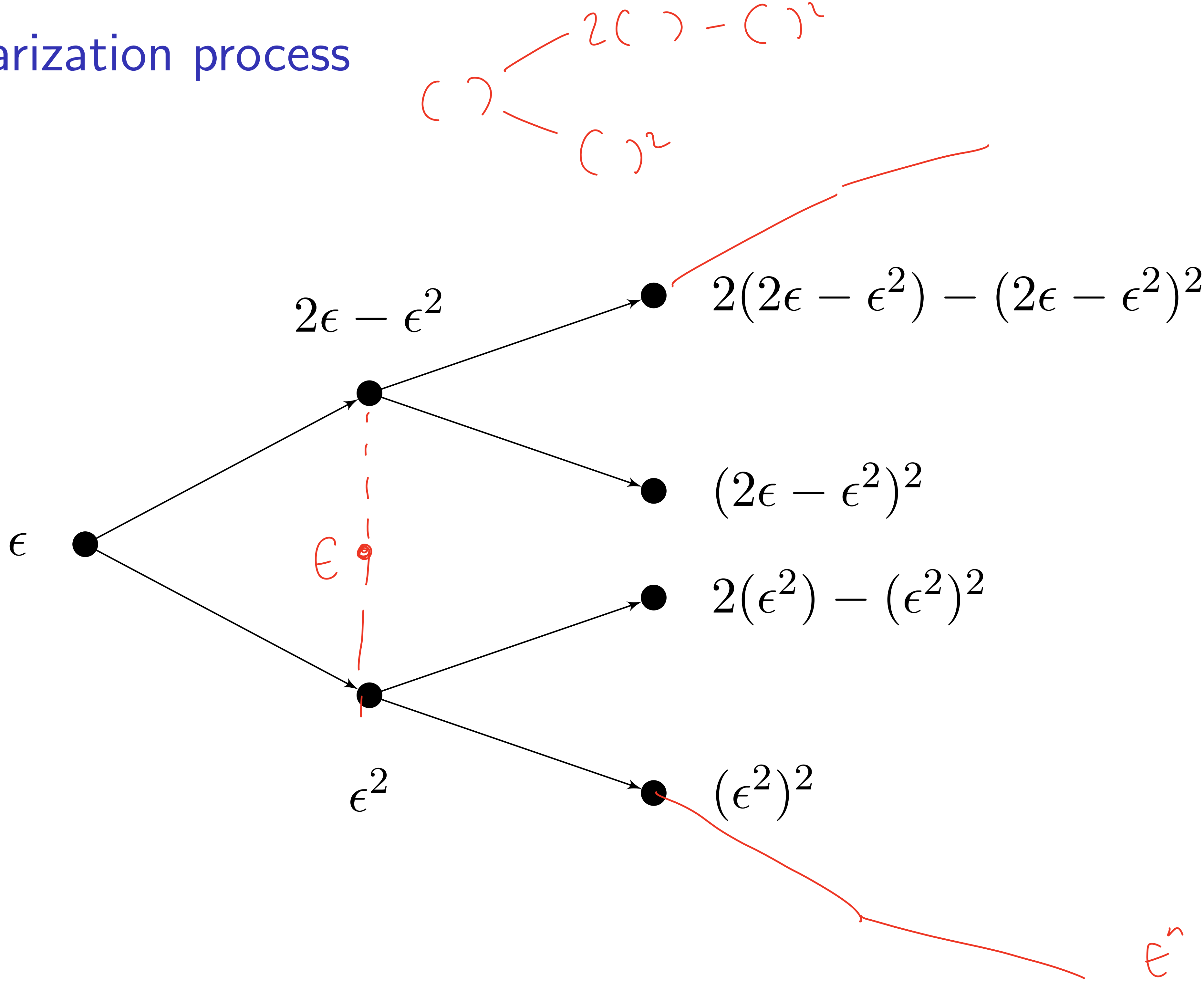
  failure probability$= 2\tilde{\epsilon} - \tilde{\epsilon}^2 = 2(\epsilon^2) - (\epsilon^2)^2$

- Decode U_4

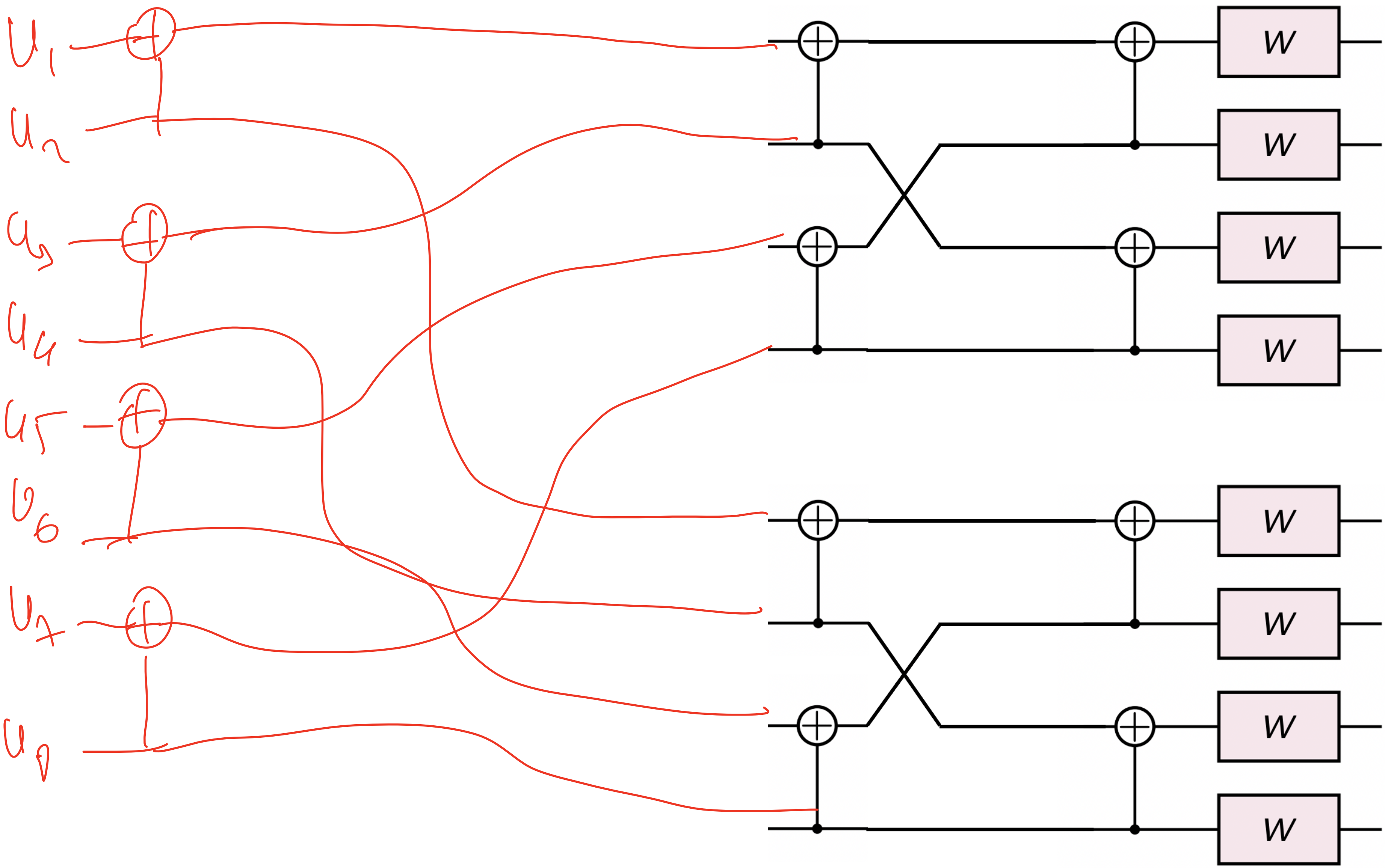
  erased if (Y_1 and Y_2 erased) **and** (Y_3 and Y_4 erased)

  failure probability$= \tilde{\epsilon}^2 = (\epsilon^2)^2$

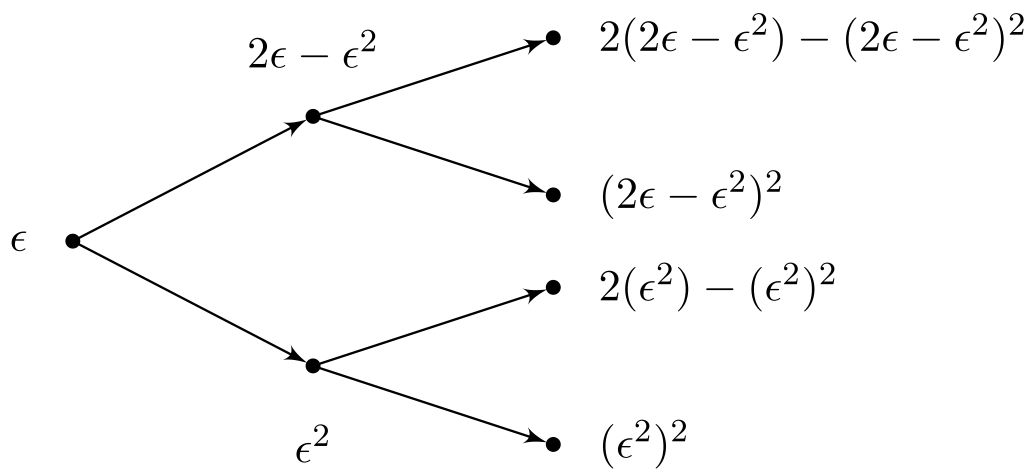
Polarization process



Larger construction

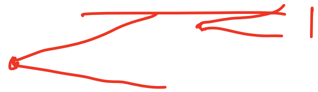


Larger construction



Question: **What happens if we keep extending the size?**

Larger construction



Gambling and Martingales

- you can bet **red** or **black**

  Probability[**red**] $= \frac{1}{2}$ Probability[**black**] $= \frac{1}{2}$
- a betting strategy that always wins (!)[1]: double the bet after every loss

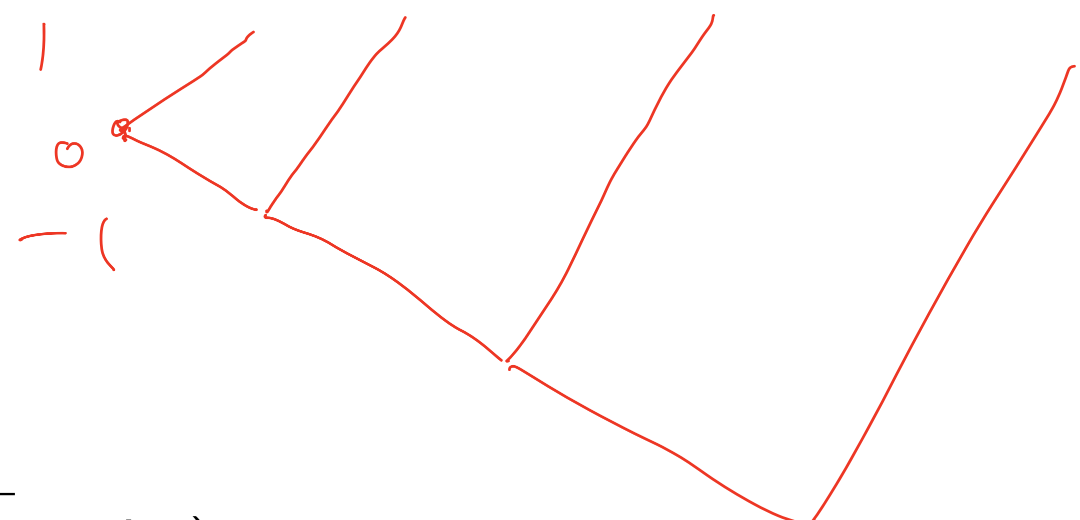
  until you win:

  bet \$1 on **black**

  bet \$2 on **black**

  bet \$4 on **black**

  ⋮



[1]do not try this at home (or at the casino)

Gambling and Martingales

until you win:

bet \$1 on **black**

bet \$2 on **black**

bet \$4 on **black**

bet \$8 on **black**

bet \$16 on **black**

bet \$32 on **black**

$\vdots$

- Martingale betting strategy is a winning strategy only if you have **unbounded wealth**
- It is **not sustainable**

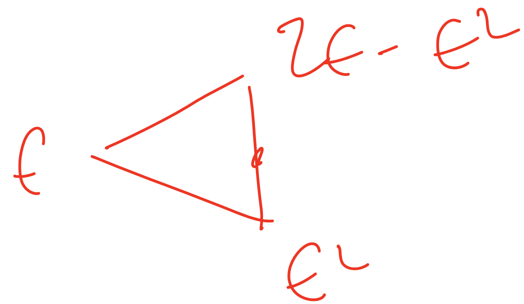
  Probability[Loosing 6 in a row]$=\frac{1}{2^6} \approx 0.016$. This will eventually happen if you repeat many times

Martingale Processes

A martingale process is a sequence of random variables for which the conditional expectation of the next value in the sequence is equal to the present value, regardless of all prior values.

- Martingales processes are important finance, e.g., in stock trading, Black Scholes option pricing model (which won the Nobel prize in economics)

Back to Polar Codes



Let e_t be random ± 1 for $t = 1, 2 \ldots$. The polarization process is

$$w_{t+1} = w_t + e_t w_t (1 - w_t)$$

$$= \begin{cases} 2w_t - w_t^2 & e_t = +1 \\ w_t^2 & e_t = -1 \end{cases}$$

Sample paths

Martingales

$$w_{t+1} = w_t + e_t w_t (1 - w_t)$$

- the expectation of the next value is equal to the previous value: $\mathbb{E}[w_{t+1}|w_t] = w_t$

Martingales

$$w_{t+1} = w_t + e_t w_t (1 - w_t)$$

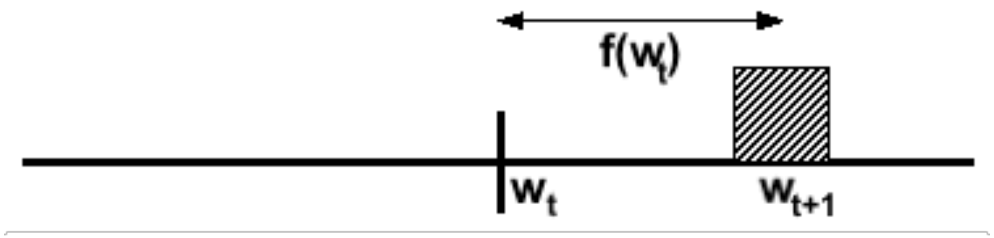
- the expectation of the next value is equal to the previous value: $\mathbb{E}[w_{t+1}|w_t] = w_t$
- Martingale processes converge to a limiting distribution if they are bounded
- polarization process $w_{t+1} = w_t + e_t w_t (1 - w_t)$ converges to $w(1-w) = 0$ **why?**
  $w = 0$ (erasure probability one) or
  $w = 1$ (erasure probability zero)

w = 0 or

w = 1

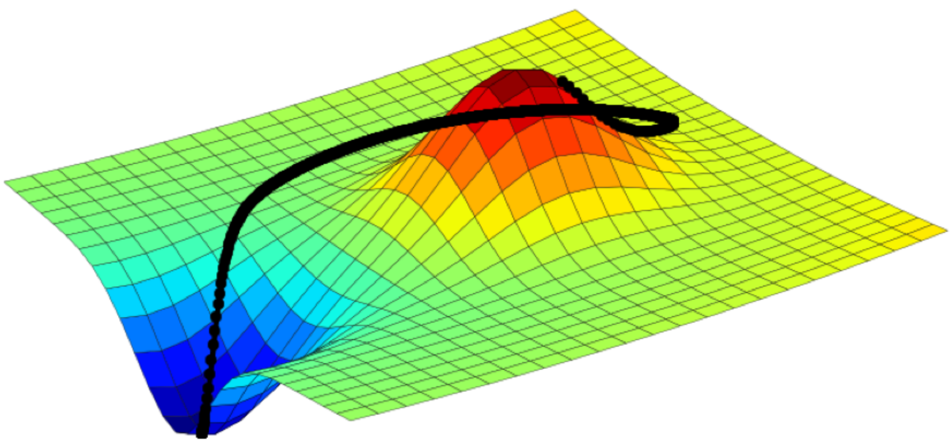
Evolution of physical systems

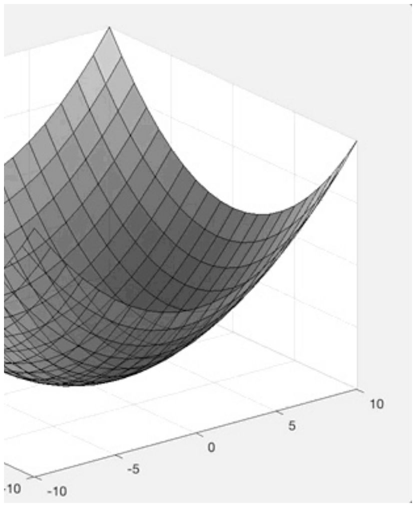
$$\underbrace{w_{t+1}}_{\text{next position}} = \underbrace{w_t}_{\text{current position}} + \underbrace{f(w_t)}_{\text{displacement}}$$



Gradient descent

$$\underbrace{w_{t+1}}_{\text{next parameters}} = \underbrace{w_t}_{\text{current parameters}} + \underbrace{f(w_t)}_{\text{-gradient}}$$



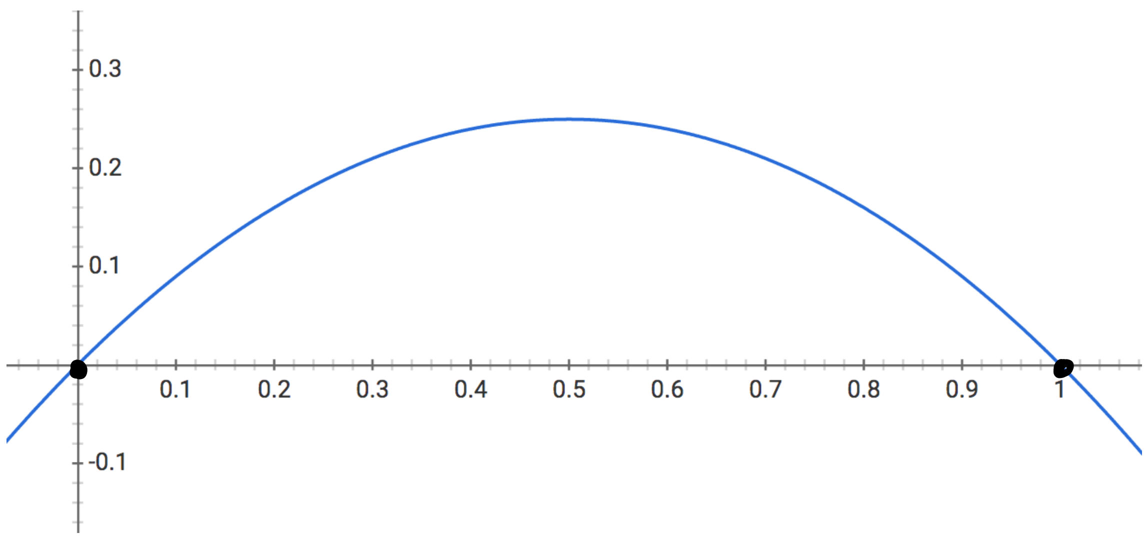


$$w_{t+1} = w_t + e_t w_t(1 - w_t)$$

converges, i.e., $w_{t+1} = w_t$ when $f(w_t) = w_t(1 - w_t) = 0$

$$w_{t+1} = w_t + e_t w_t(1 - w_t)$$

plot of $f(w) = w(1 - w)$



Polarization theorem

- the process

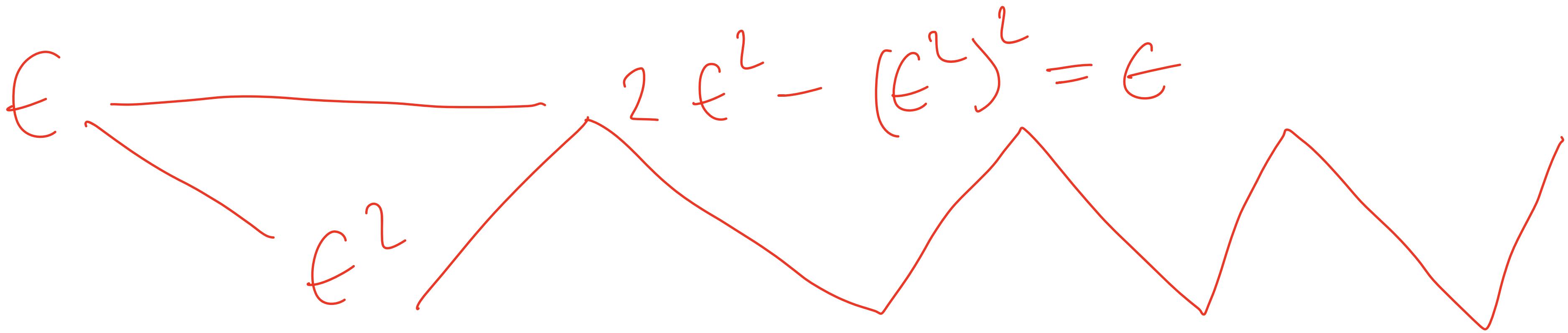
$$w_{t+1} = w_t + e_t w_t (1 - w_t)$$

  converges to either zero or one with probability one!
- implies that almost all channels are either perfect or completely noisy

Non-convergent paths

- Down - Up - Down - Up ....

  $\epsilon \searrow \epsilon^2 \nearrow 2\epsilon^2 - \epsilon^4 =? \epsilon$



Non-convergent paths

- Down - Up - Down - Up ....

  $\epsilon \searrow \epsilon^2 \nearrow 2\epsilon^2 - \epsilon^4 = \epsilon$ if $\epsilon = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{1}{\phi} \approx 0.61803398875$

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$$\text{Golden ratio :} \quad \phi := \frac{1+\sqrt{5}}{2} \approx 1.61803398875$$

Non-convergent paths

- Down - Up - Down - Up ....

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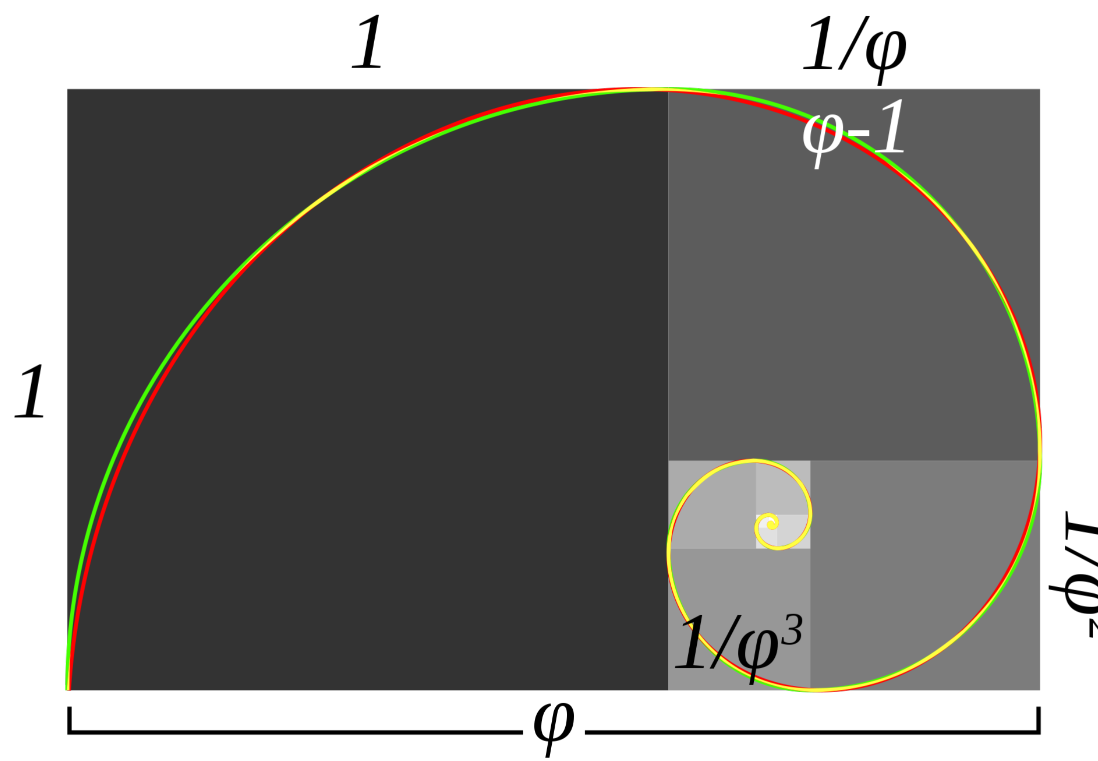
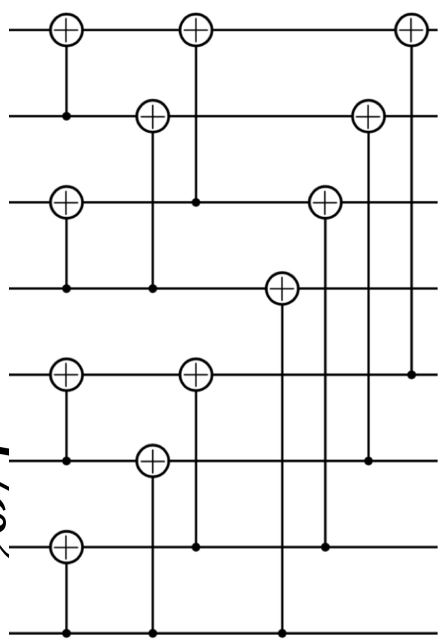
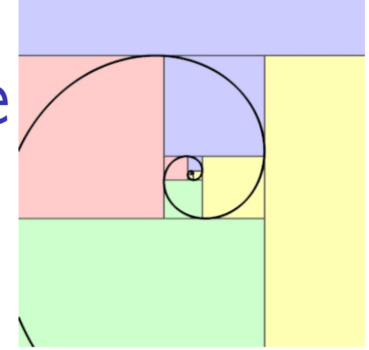
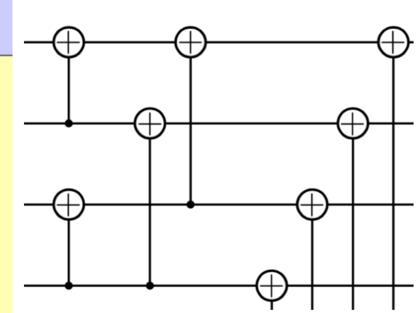
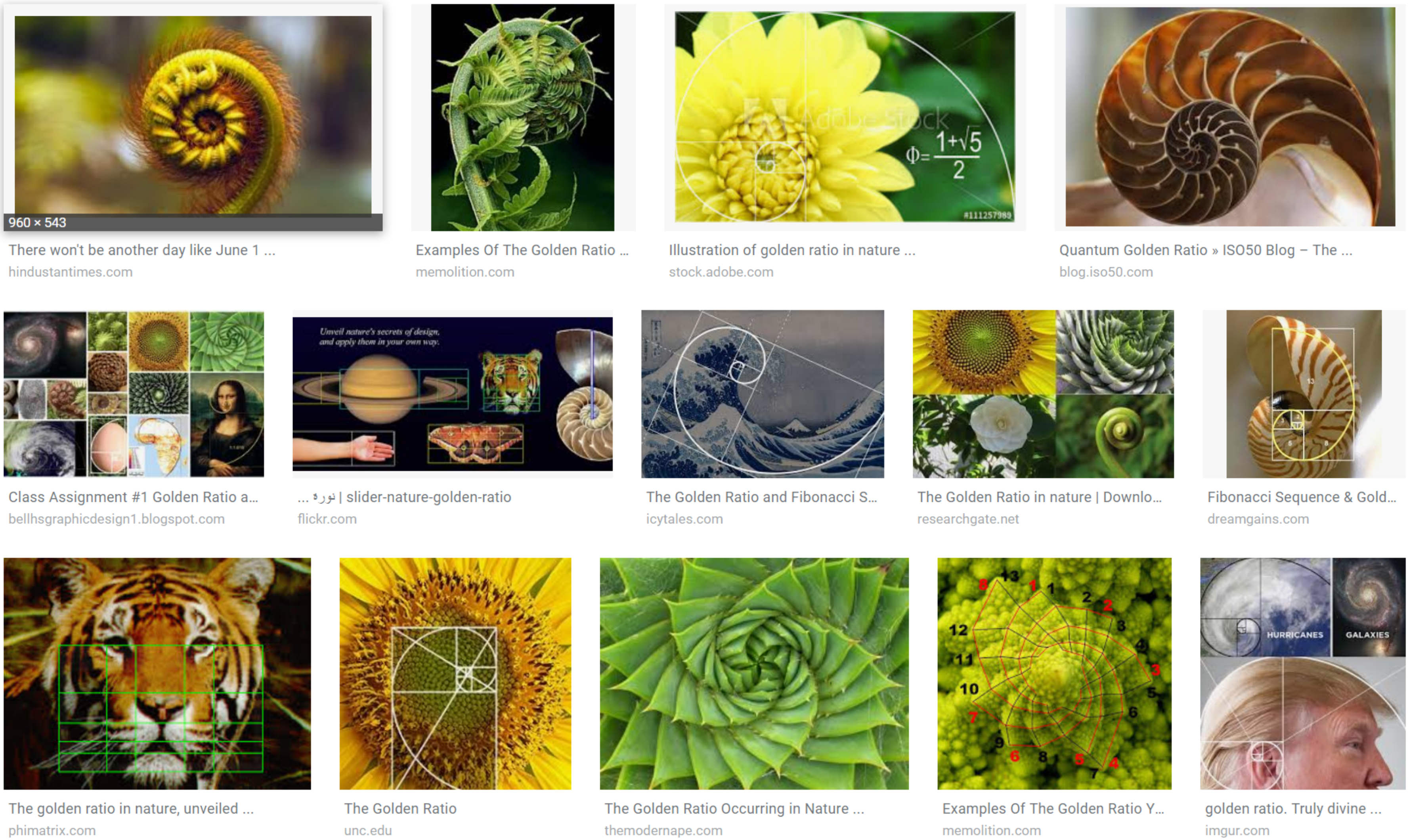




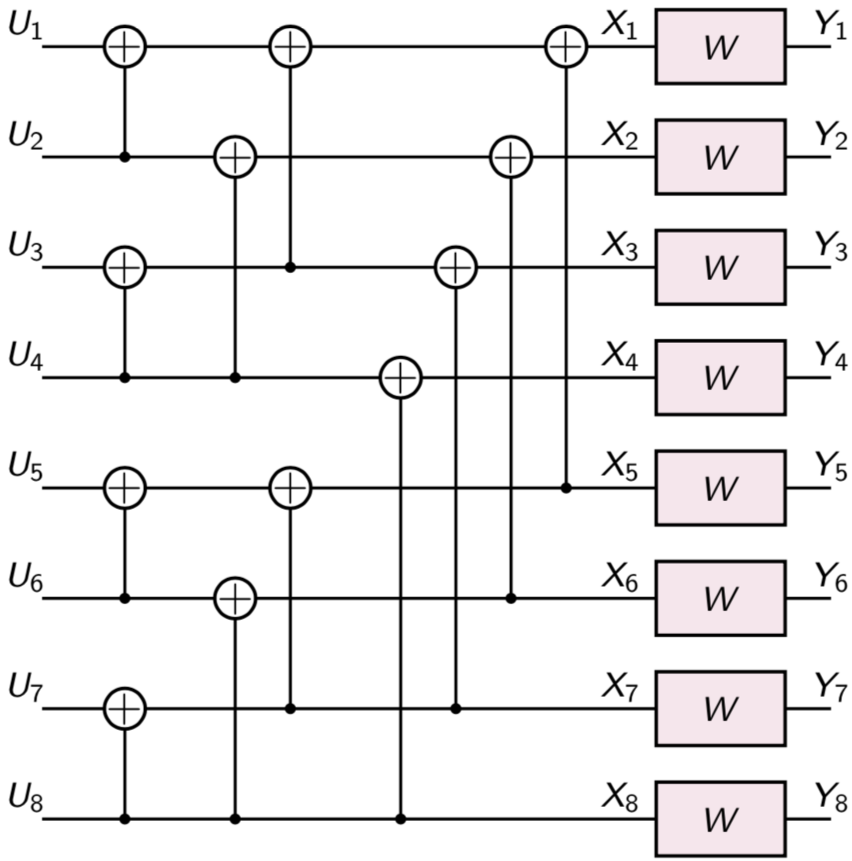
Google images: golden ratio in nature







Encoding circuit



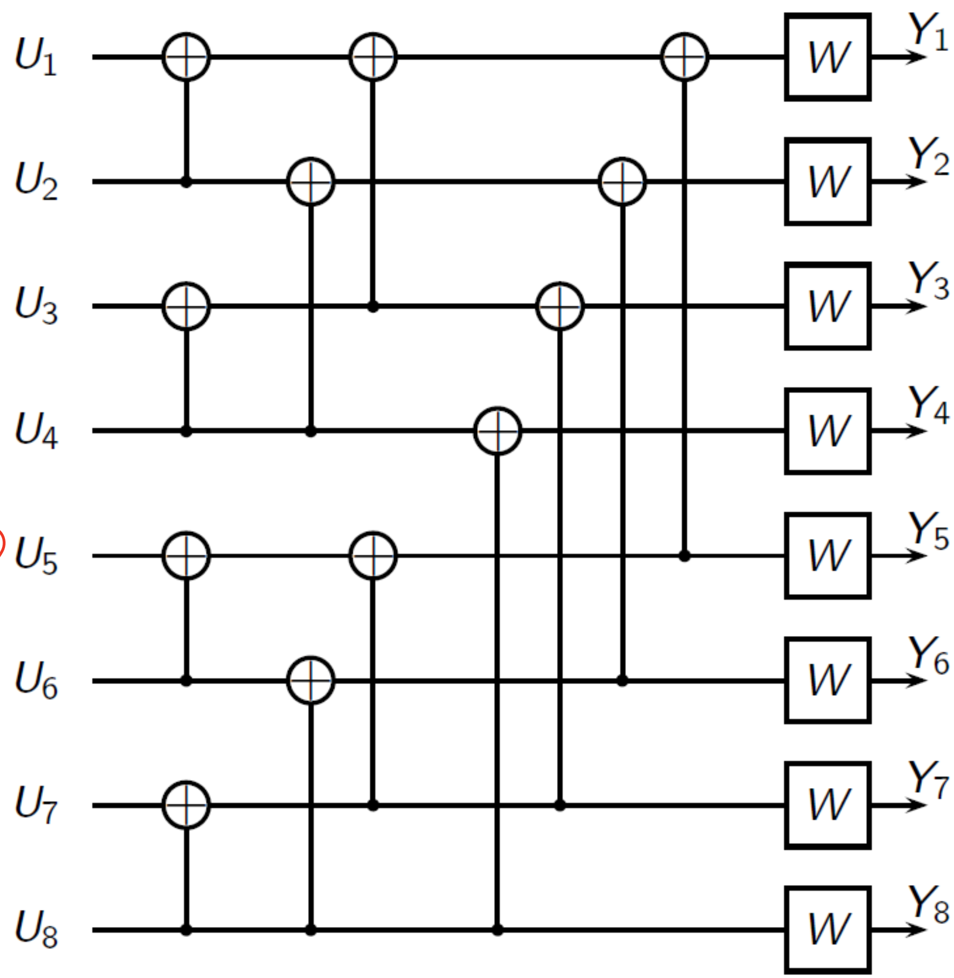
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

What do we do with the noisy channels?

Rate = $\frac{1}{2}$

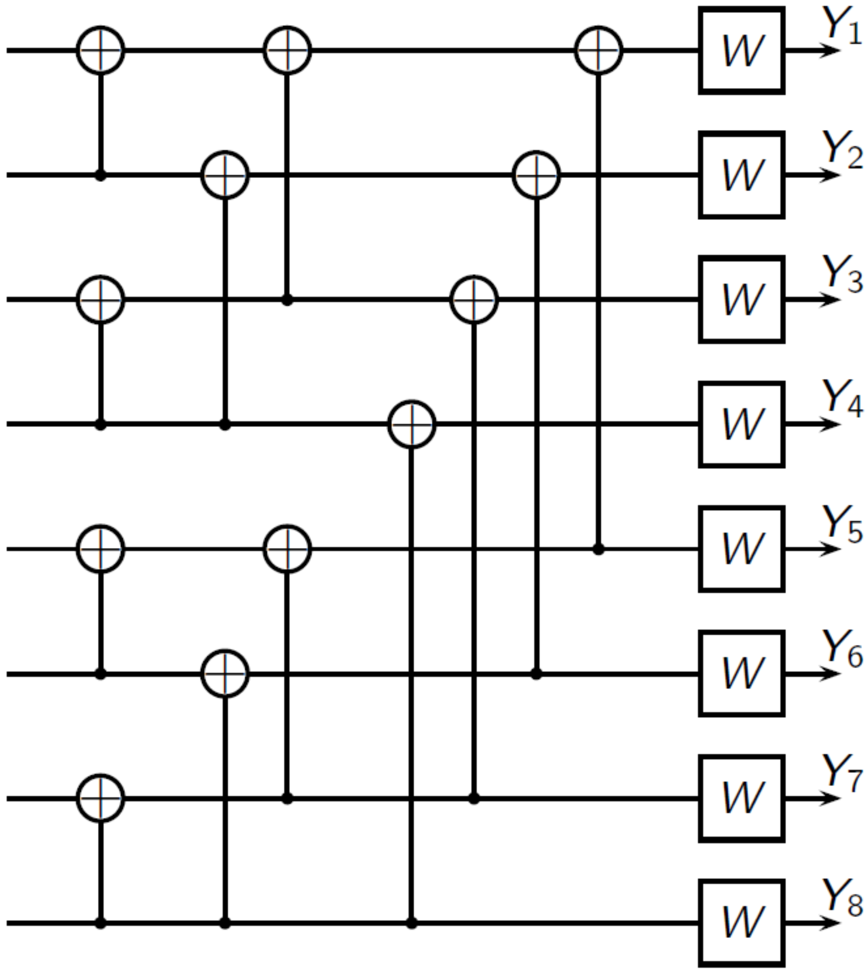
C=1-P[erasure]	rank
0.0039	8
0.1211	7
0.1914	6
✓ 0.6836	4
0.3164	5
✓ 0.8086	3
✓ 0.8789	2
✓ 0.9961	1

$u_1 = 0$
$u_2 = 0$
$u_3 = 0$
$u_5 = 0$



We can freeze noisy channels!

C=1-P[erasure]	rank		
0.0039	8	frozen	U_1
0.1211	7	frozen	U_2
0.1914	6	frozen	U_3
0.6836	4	data	U_4
0.3164	5	frozen	U_5
0.8086	3	data	U_6
0.8789	2	data	U_7
0.9961	1	data	U_8



Freezing noisy channels

C=1-P[erasure]	rank		
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0.8086	3	data	U_6
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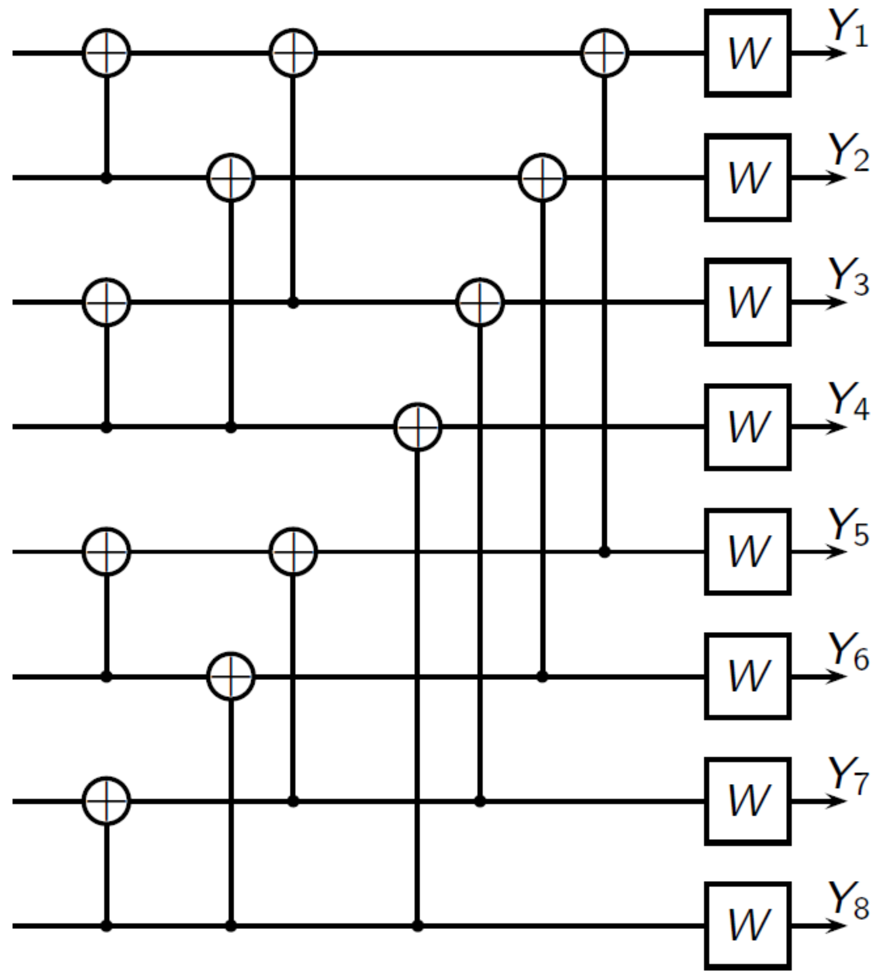
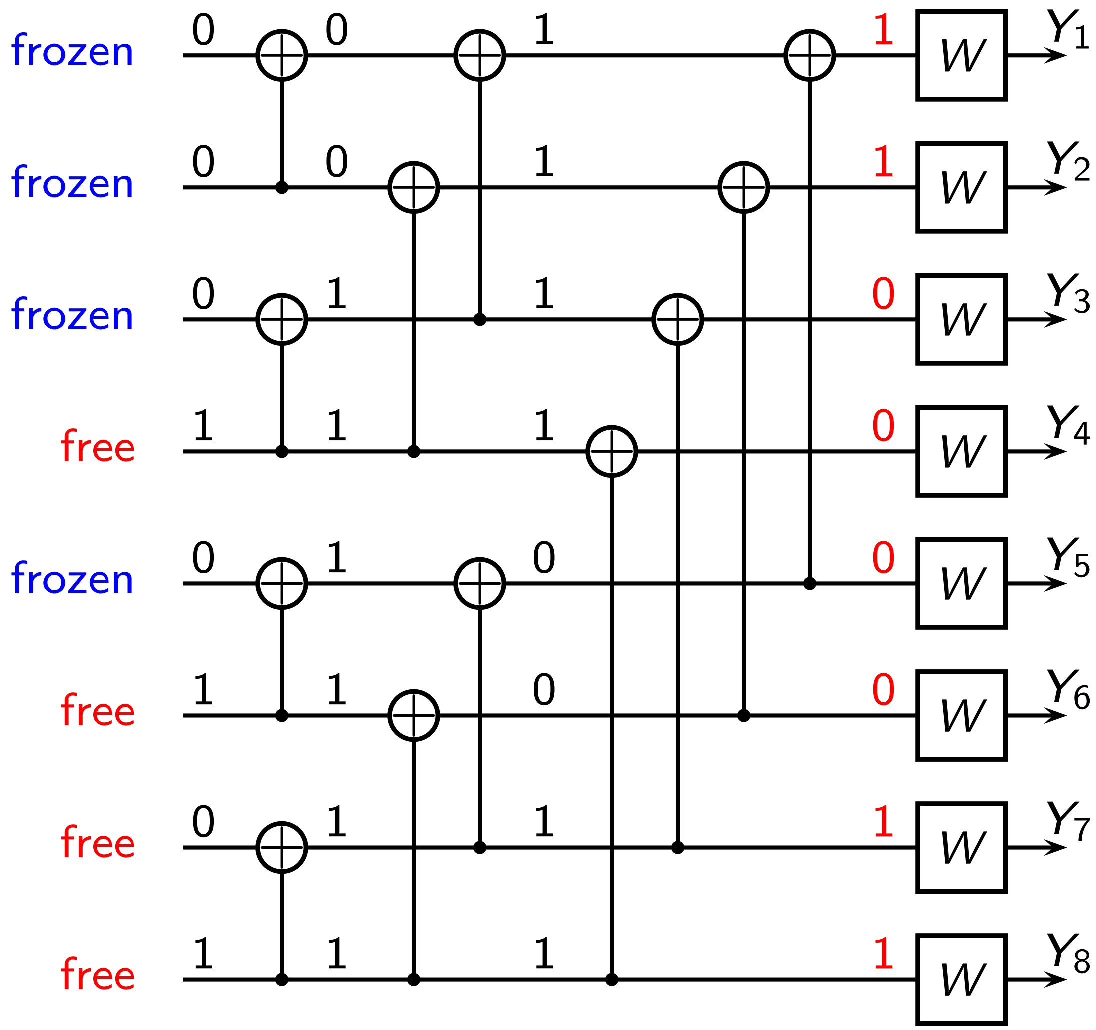


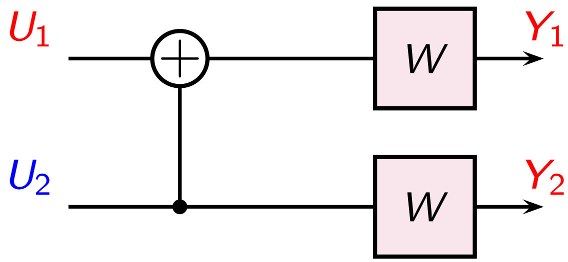
Encoding and decoding

all the bad channels are frozen

- successive cancellation decoder will correctly recover the message with high probability!



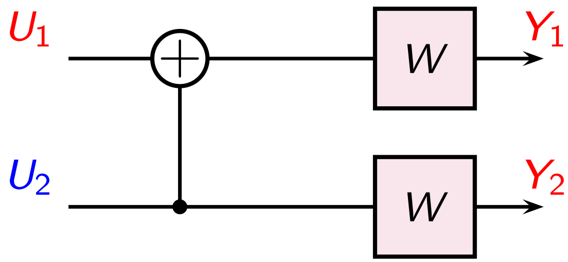
Polarization of general channels



$$W^{-}(Y_1, Y_2 | U_1) = \frac{1}{2} \sum_{u_2} W_1(y_1 | u_1 \oplus u_2) W_2(y_2 | u_2)$$

$$W^{+}(Y_1, Y_2, U_1 | U_2) = \frac{1}{2} W_1(y_1 | u_1 + u_2) W_2(y_2 | u_2)$$

Polarization of general channels



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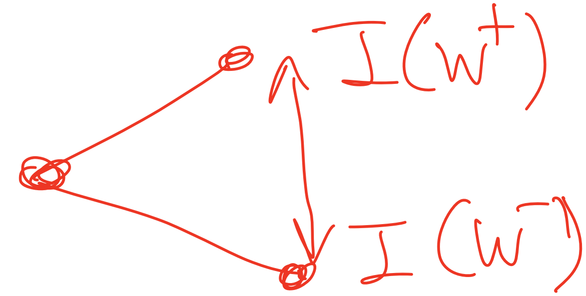
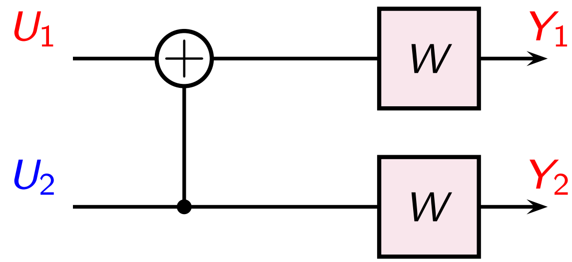
$$W^+(Y_1, Y_2, U_1|U_2) = \frac{1}{2} W_1(y_1|u_1 + u_2)W_2(y_2|u_2)$$

$$I(W^-) + I(W^+) = I(W) + I(W) = 2I(W)$$

Polarization of general channels





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$$I(W^-) + I(W^+) = I(W) + I(W) = 2I(W)$$

2p(1-p)=p
p=0 or 1/2

Mrs Gerber's Lemma: If $I(W) = 1 - \mathcal{H}(p)$, then

$$\frac{1}{2}(I(W^+) - I(W^-)) \geq \mathcal{H}(2p(1-p)) - \mathcal{H}(p) > 0$$

Polarization theorem

$C(W) = 1 - \epsilon$

$C(W)$ = capacity of the original channel

- $C(W)$ fraction of channels converge to noiseless channels with mutual information ≈ 1
- $1 - C(W)$ fraction of channels converge noisy channels with mutual information ≈ 0

$C(W)$ fraction

$C(W)$—

$1 - C(W)$ fraction

Polarization theorem

$C(W) =$ capacity of the original channel

- $C(W)$ fraction of channels converge to noiseless channels with mutual information ≈ 1
- $1 - C(W)$ fraction of channels converge noisy channels with mutual information $\approx$ 0

n total channel uses:

$nC(W)$ noiseless and $n(1 - C(W))$ noisy

- By freezing the noisy channels to zero we get $Rate \rightarrow \frac{nC(W)}{n} = C(W)$
- Achieves capacity as n gets large! This is true for any symmetric channel!

Polarization Theorem (formal)

Theorem

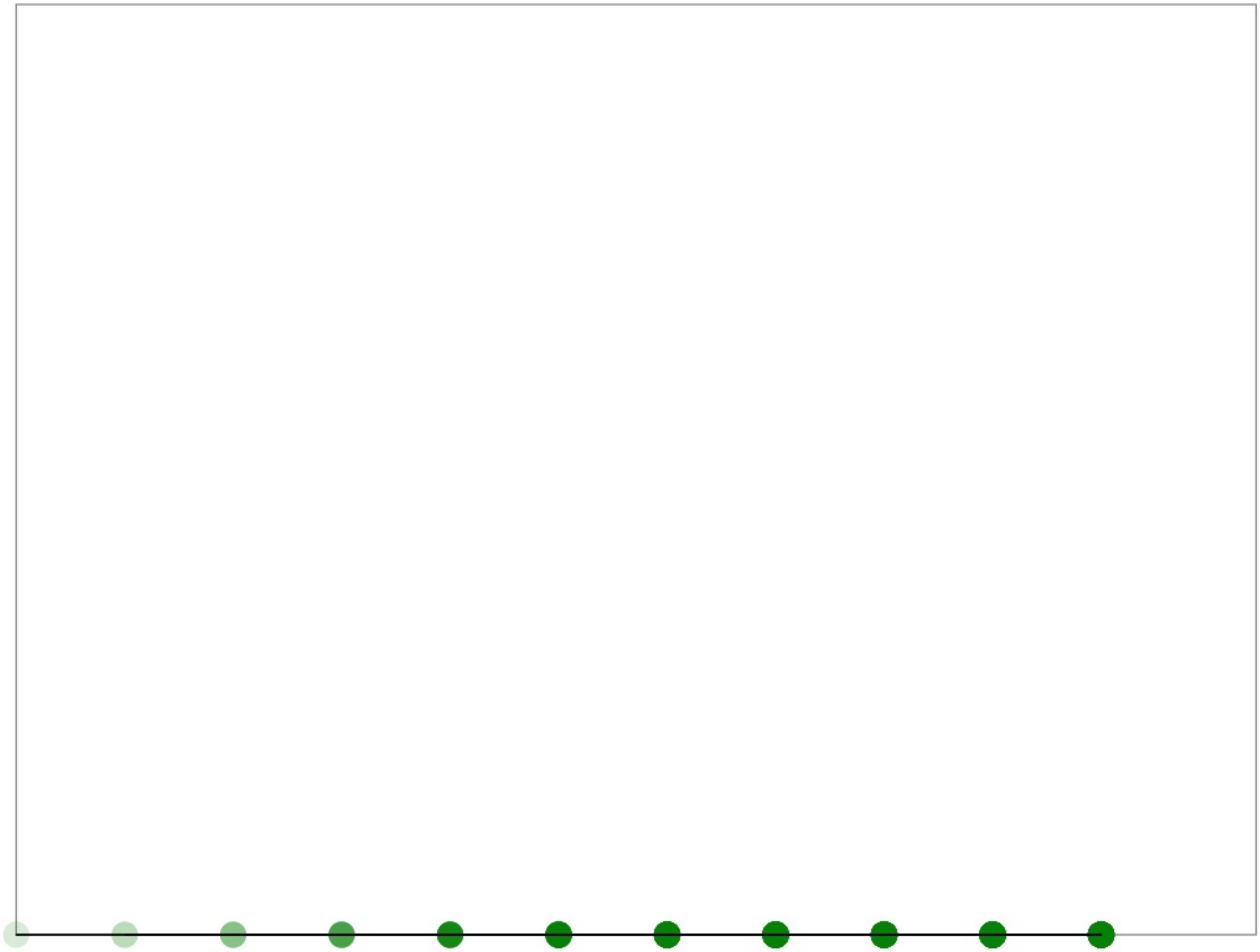
The bit-channel capacities $\{C(W_i)\}$ *polarize: for any* $\delta \in (0,1)$, *as the construction size N grows*

$$\left[\frac{\text{no. channels with } C(W_i) > 1-\delta}{N}\right] \longrightarrow C(W)$$

and

$$\left[\frac{\text{no. channels with } C(W_i) < \delta}{N}\right] \longrightarrow 1 - C(W)$$

Polarization as capacity changes



Consequence of the Polarization Theorem

Theorem

For any rate $R < I(W)$ and block-length N, the probability of frame error for polar codes under successive cancelation decoding is bounded as

$$P_e(N, R) = o\left(2^{-\sqrt{N}+o(\sqrt{N})}\right)$$

random codes

$$2^{-\frac{N}{2}}$$

5G Communications

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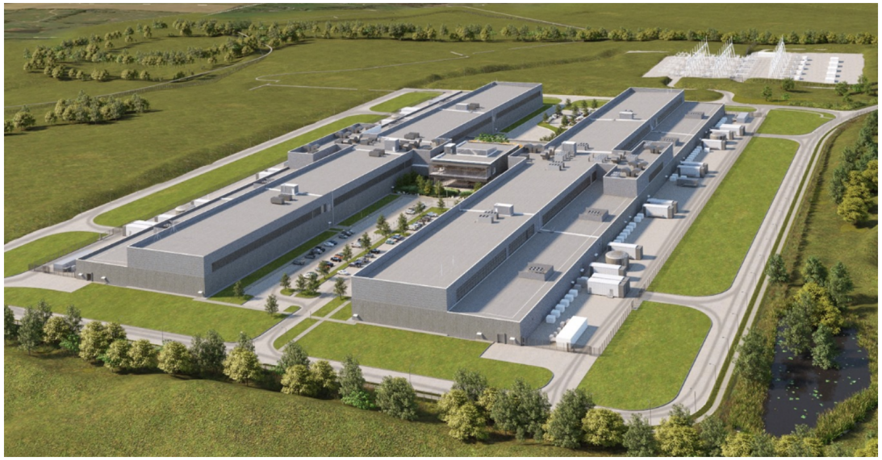
- Current LTE download speed is **5-12 Mbps**
- In November 2016, 3GPP agreed to adopt Polar codes for control channels in 5G. LDPC codes will be used in data channels.

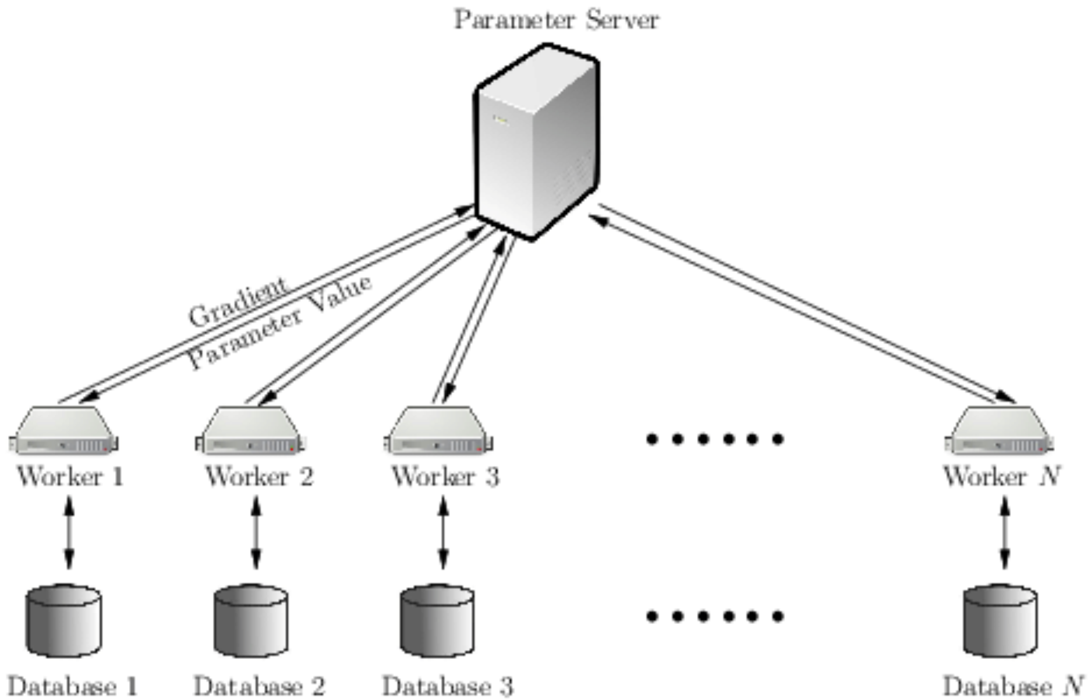
Other Applications: Distributed Computing in Data Centers



Facebook Data Center, New Albany OH.

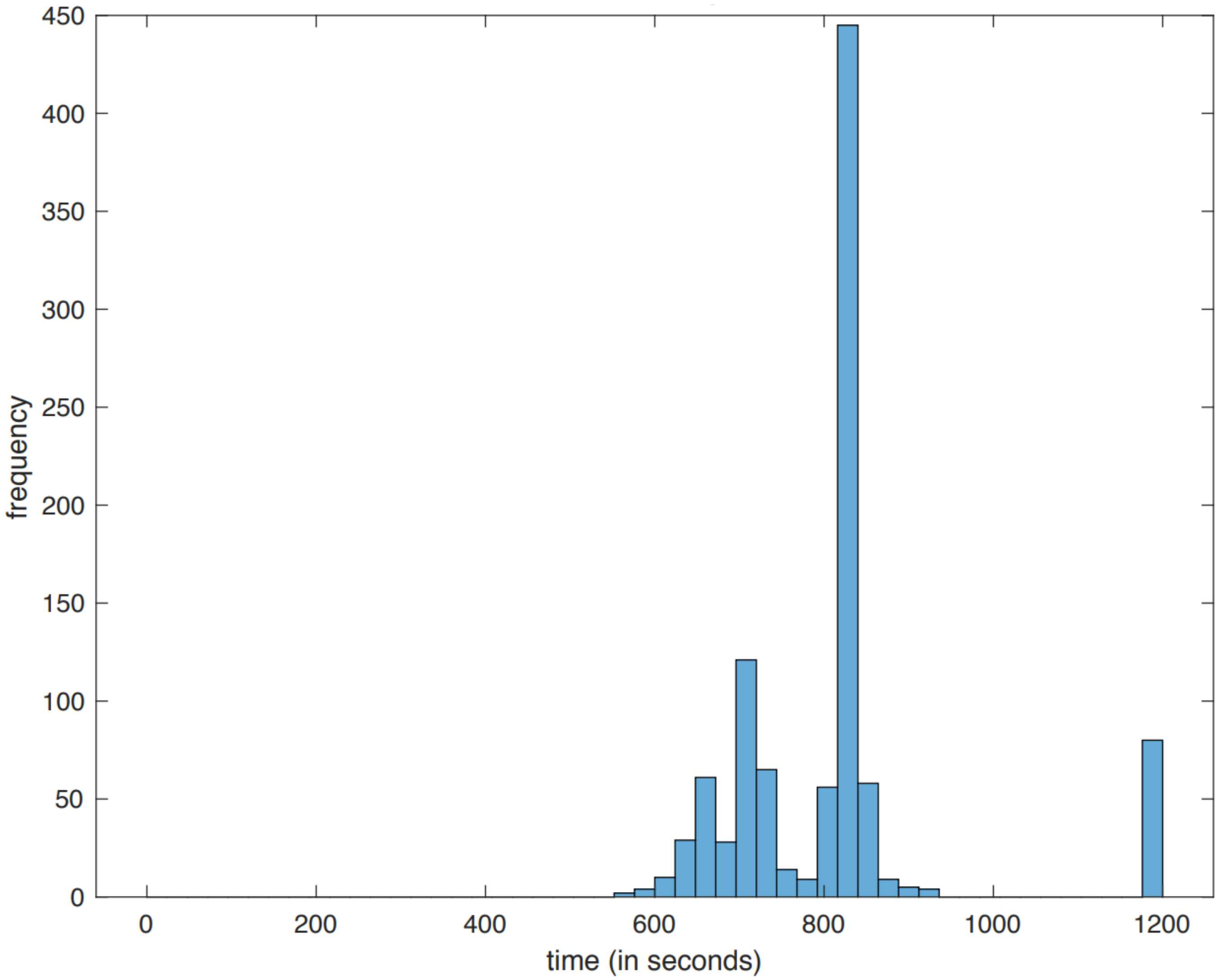
Data Centers

Distributed Computing

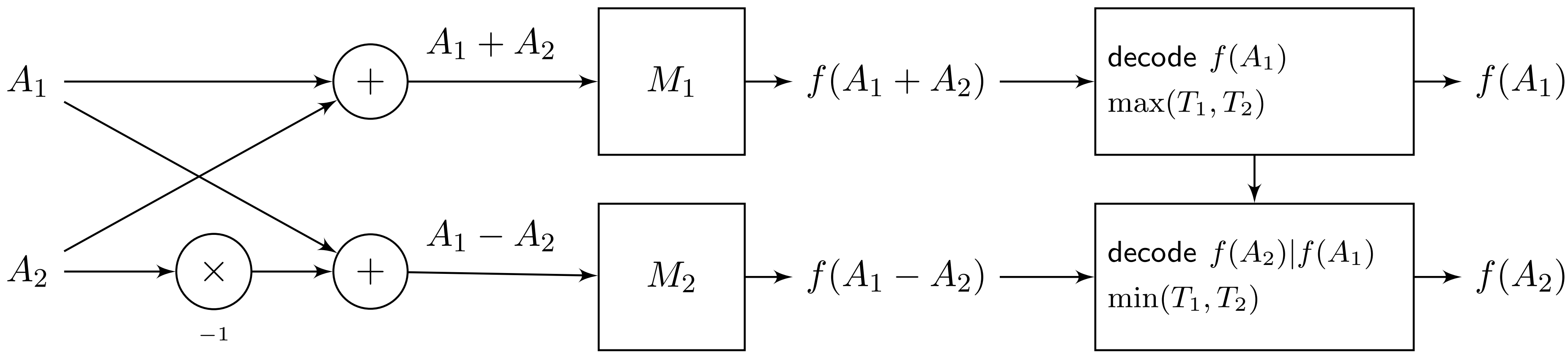


need to wait workers to finish local computations

Distributed Computing



Computational Polarization

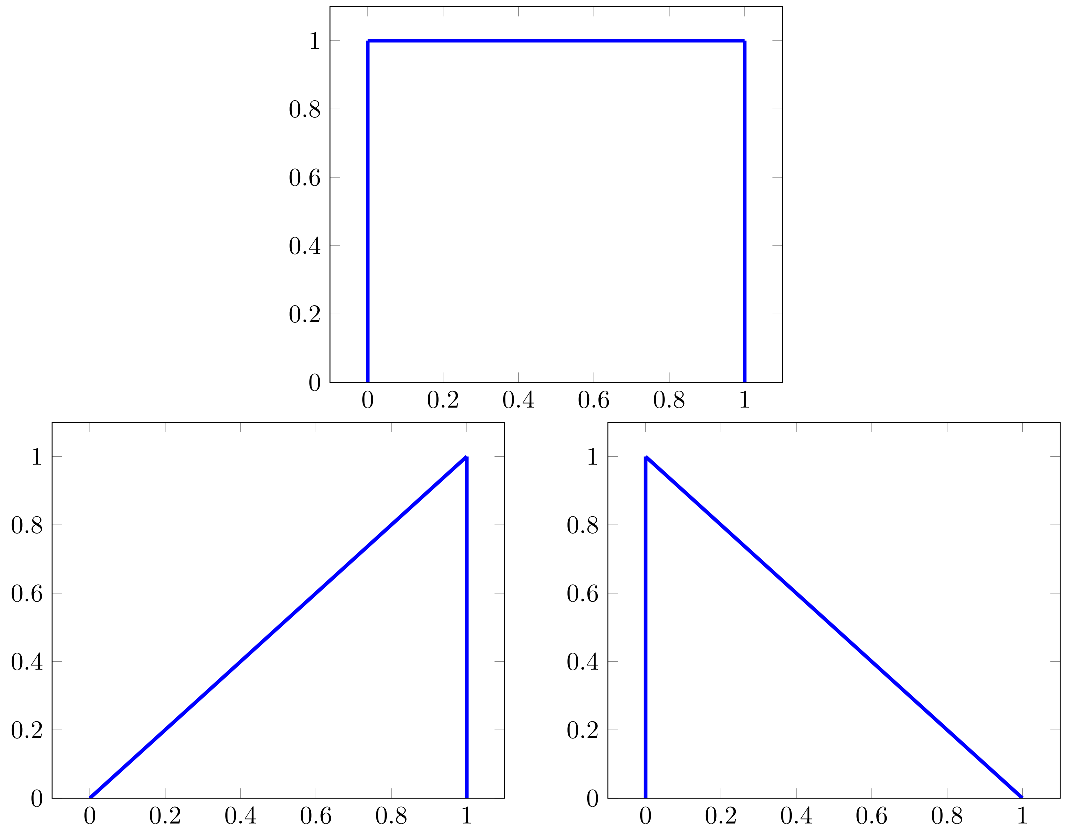


$f(A_1) = A_1 D$

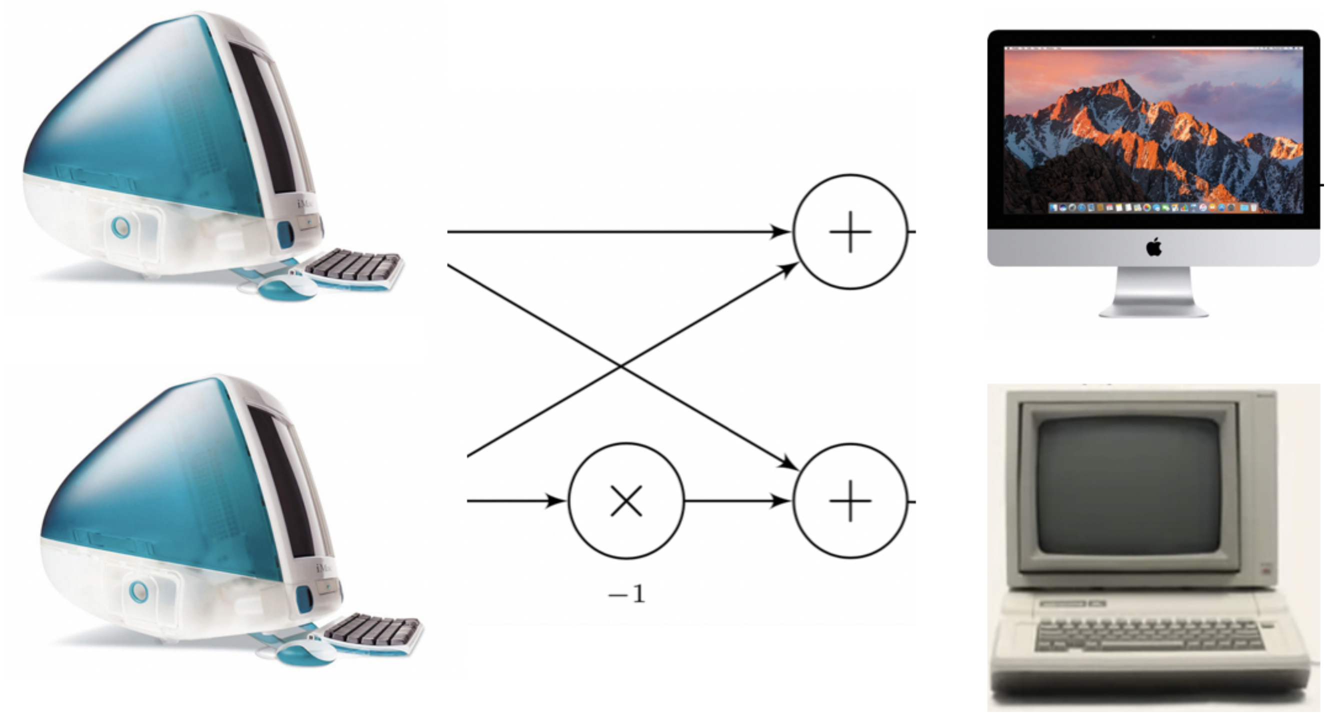
M. Pilanci, **Computational Polarization: An Information-Theoretic Method for Resilient Computing,** IEEE Transactions on Information Theory, 2022

B. Bartan and M. Pilanci, **Straggler Resilient Serverless Computing Based on Polar Codes**, Annual Allerton Conference on Communication, Control, and Computing 2019. arxiv.org/pdf/1901.06811

Polarization of computation times



Polarization for computation



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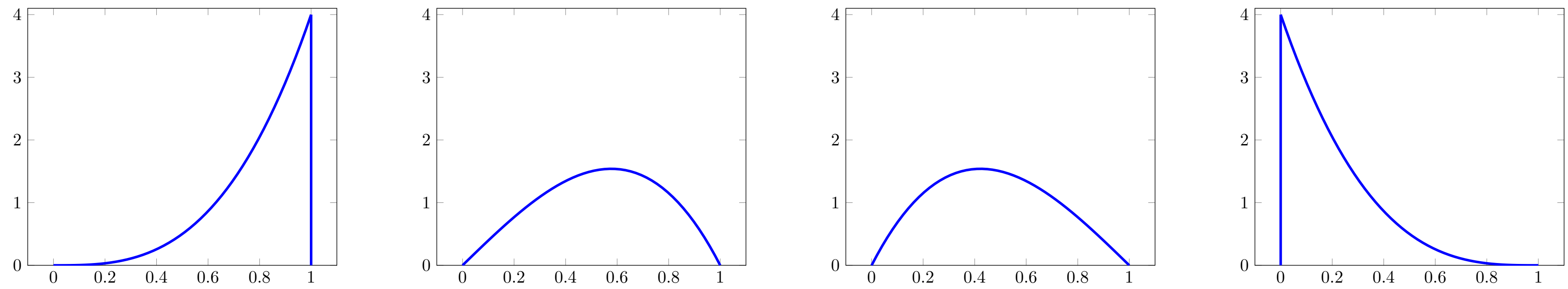
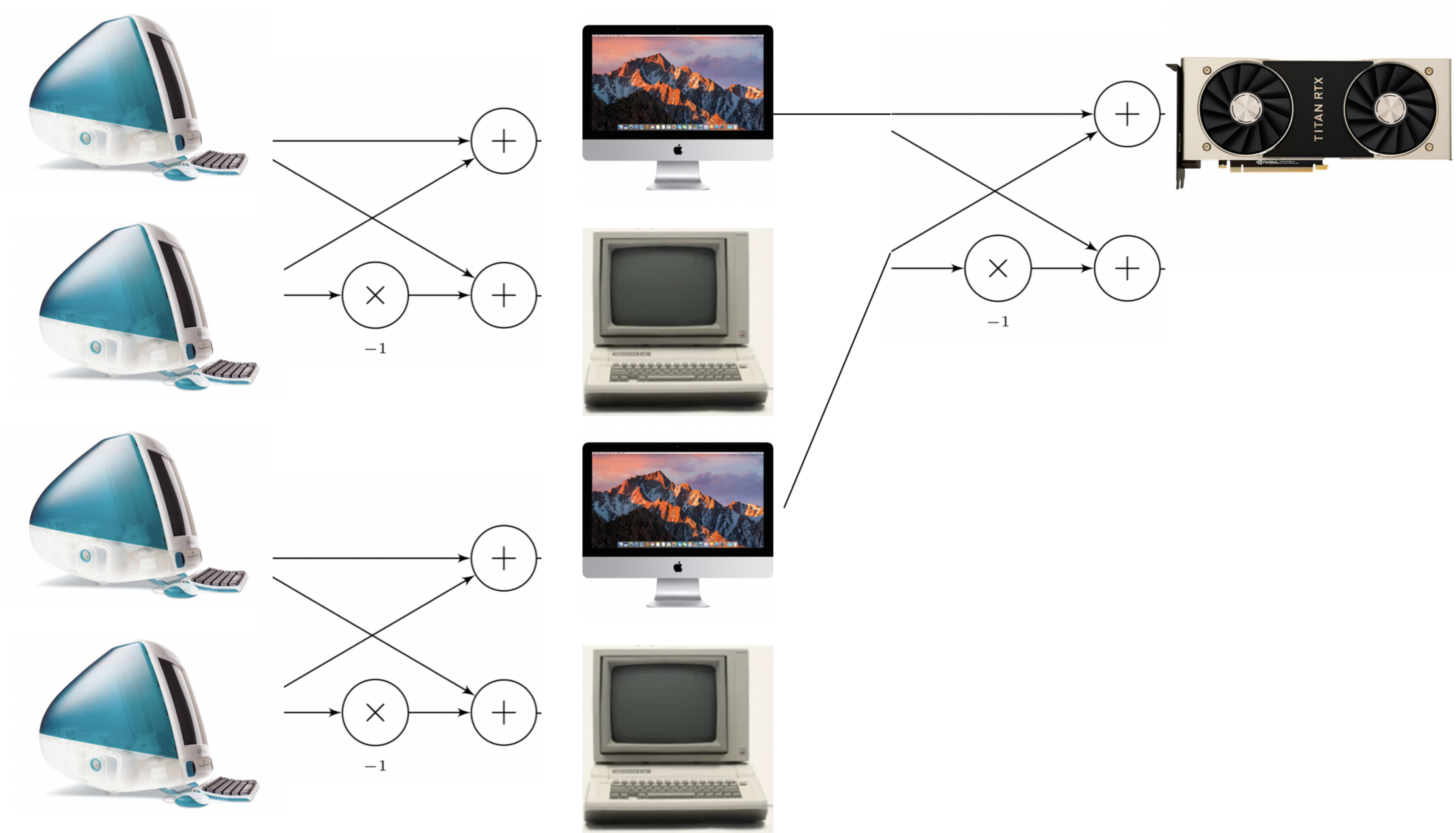


Polar codes for computation



Polar coded machine learning

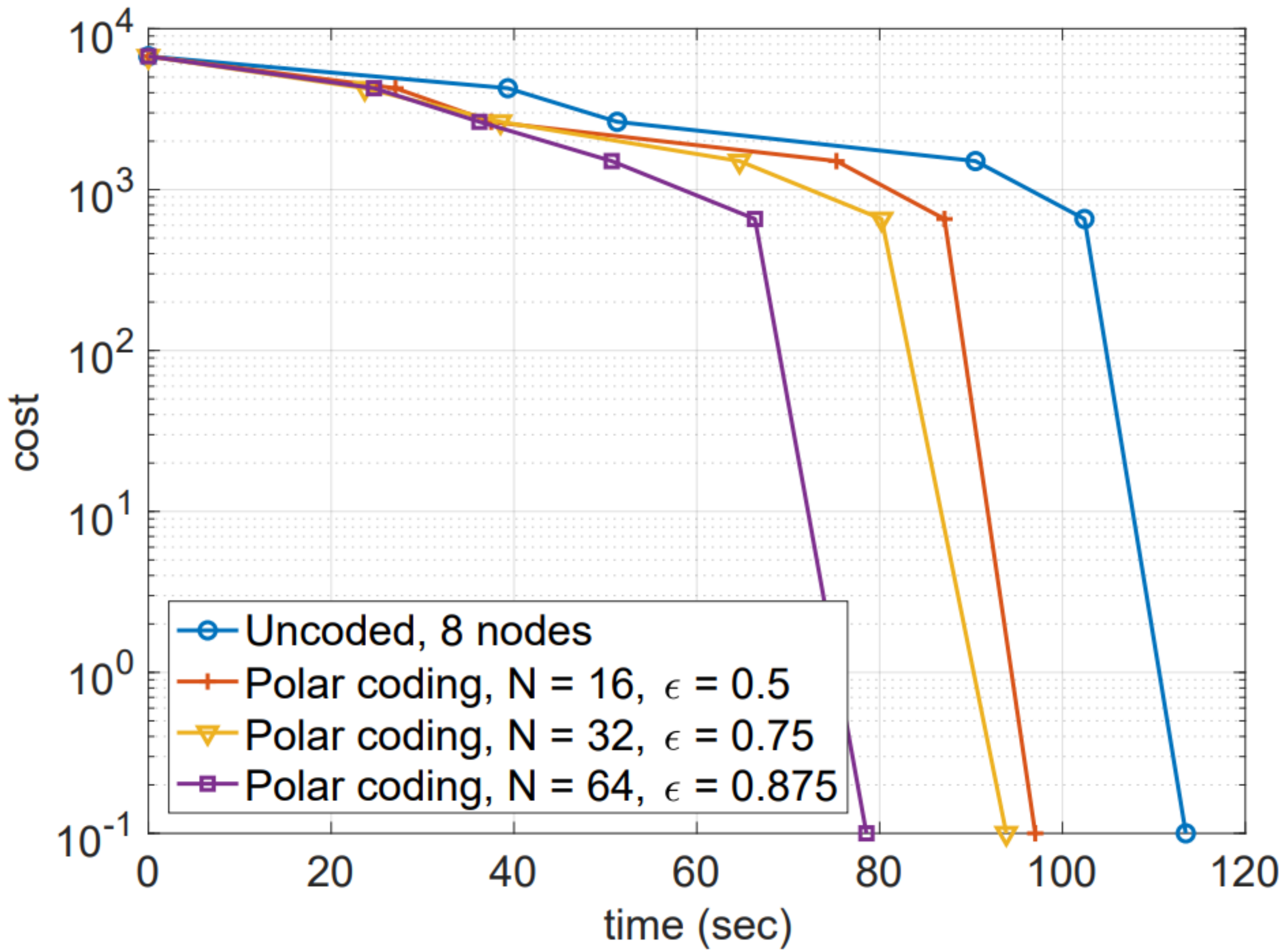


Fig. 8. Cost vs time for the gradient descent example.

B. Bartan and M. Pilanci, **Straggler Resilient Serverless Computing Based on Polar Codes**, Annual Allerton Conference on Communication, Control, and Computing 2019. arxiv.org/pdf/1901.06811

Questions?

Cloud computing on Amazon Lambda

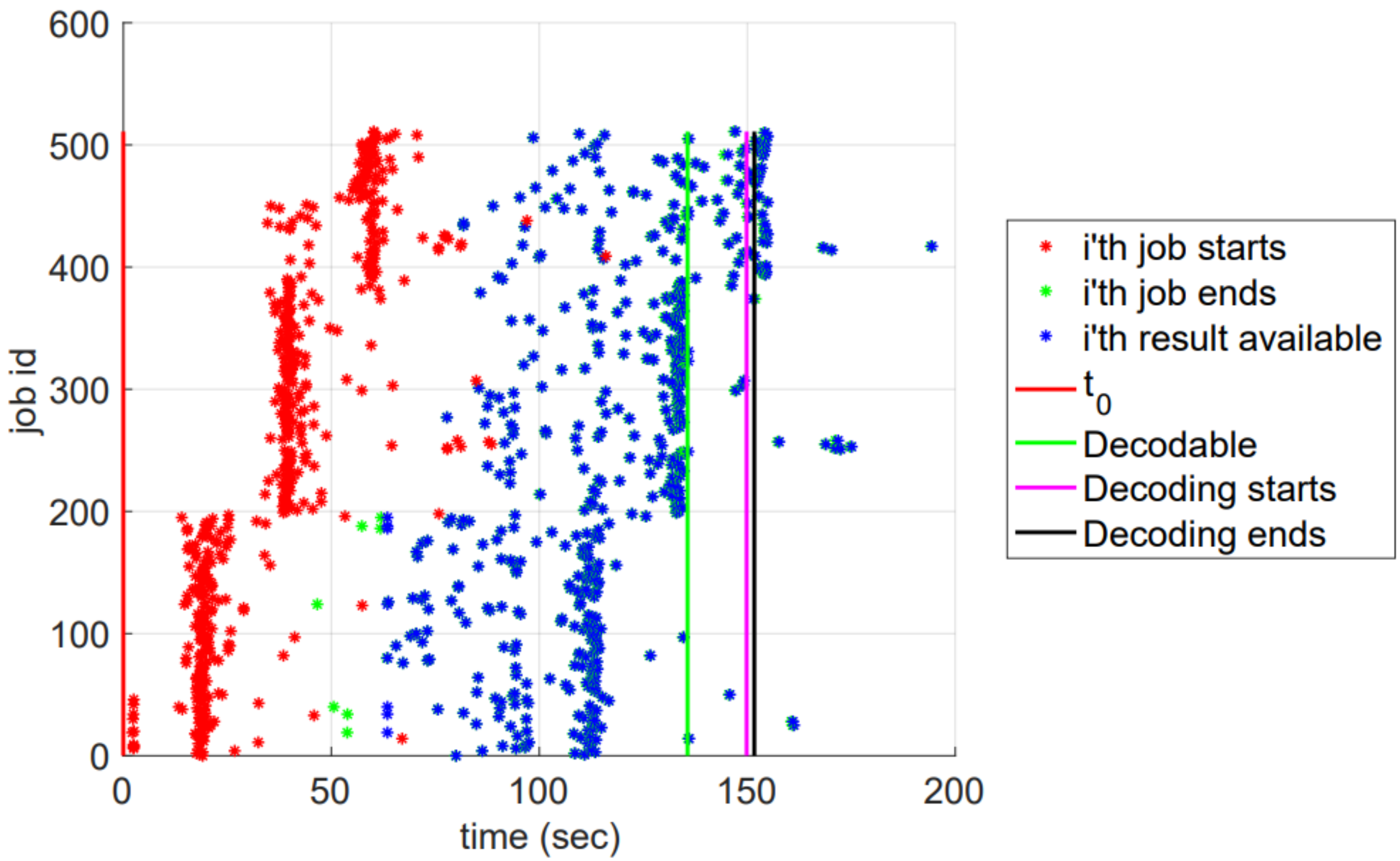
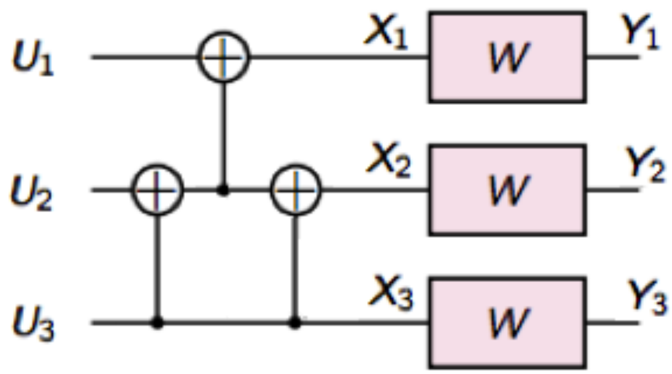


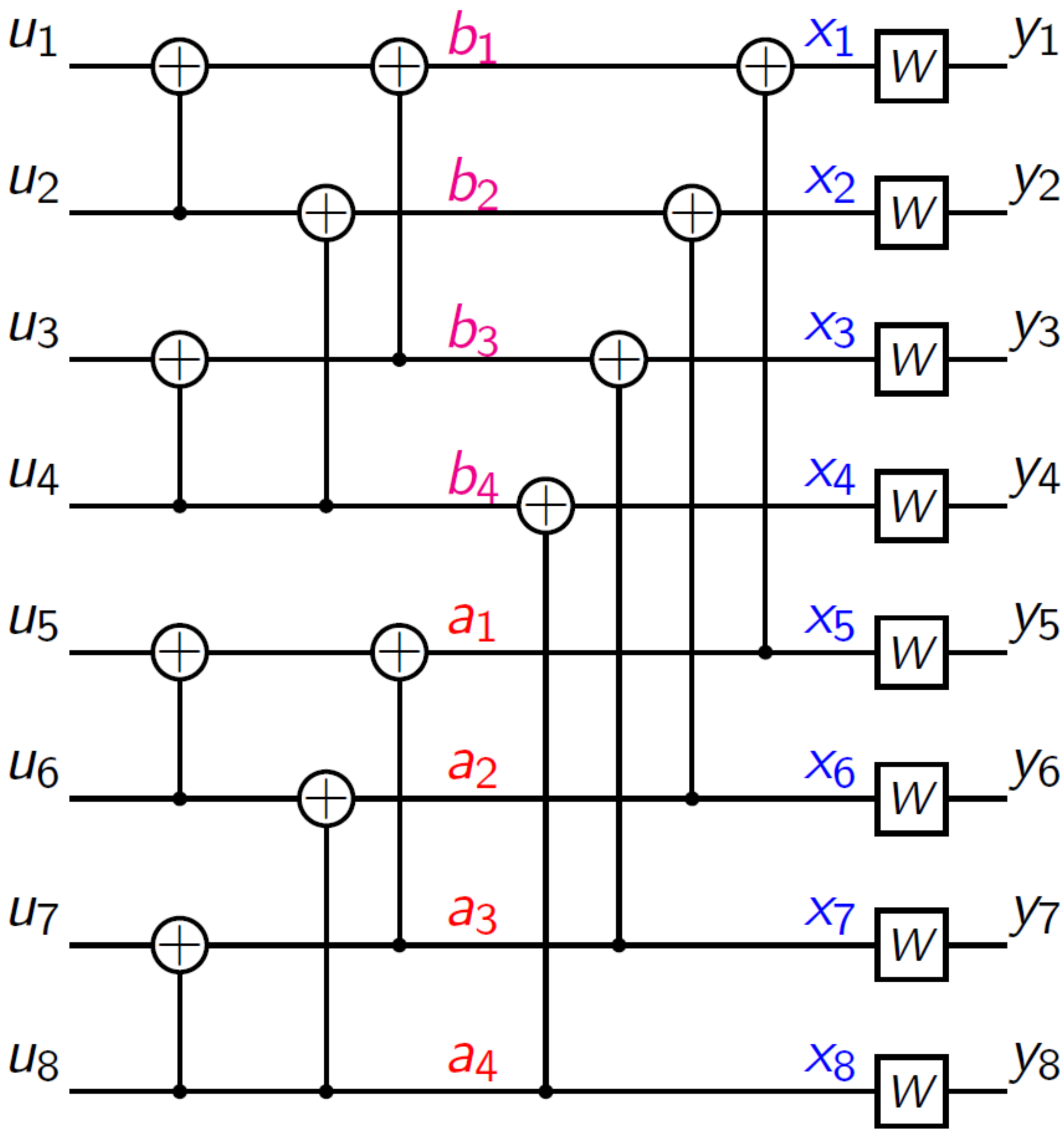
Fig. 7. Job output times and decoding times for $N = 512$.

Extensions

- Ternary Erasure Channel

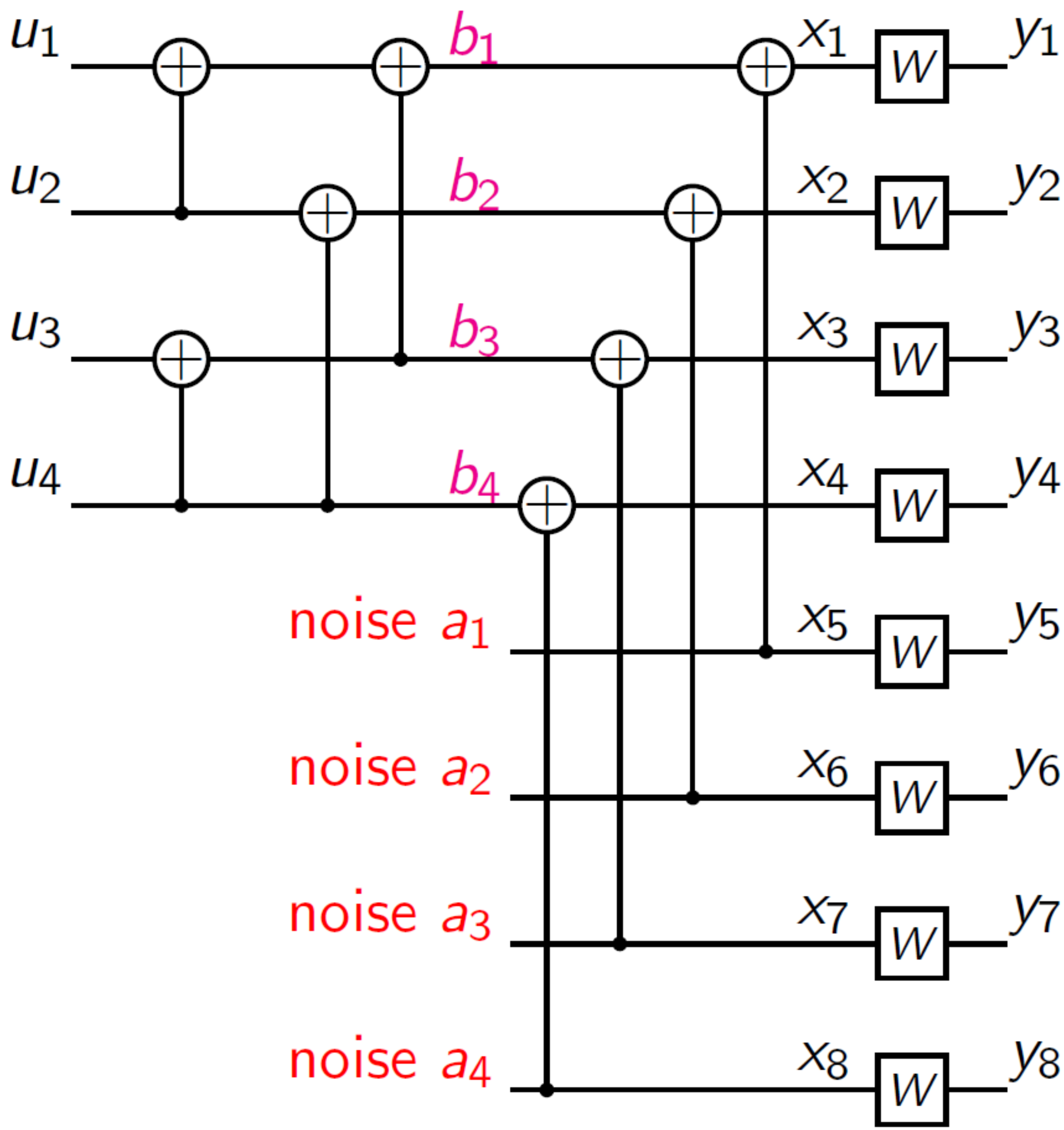


Details on Decoding: Divide and Conquer



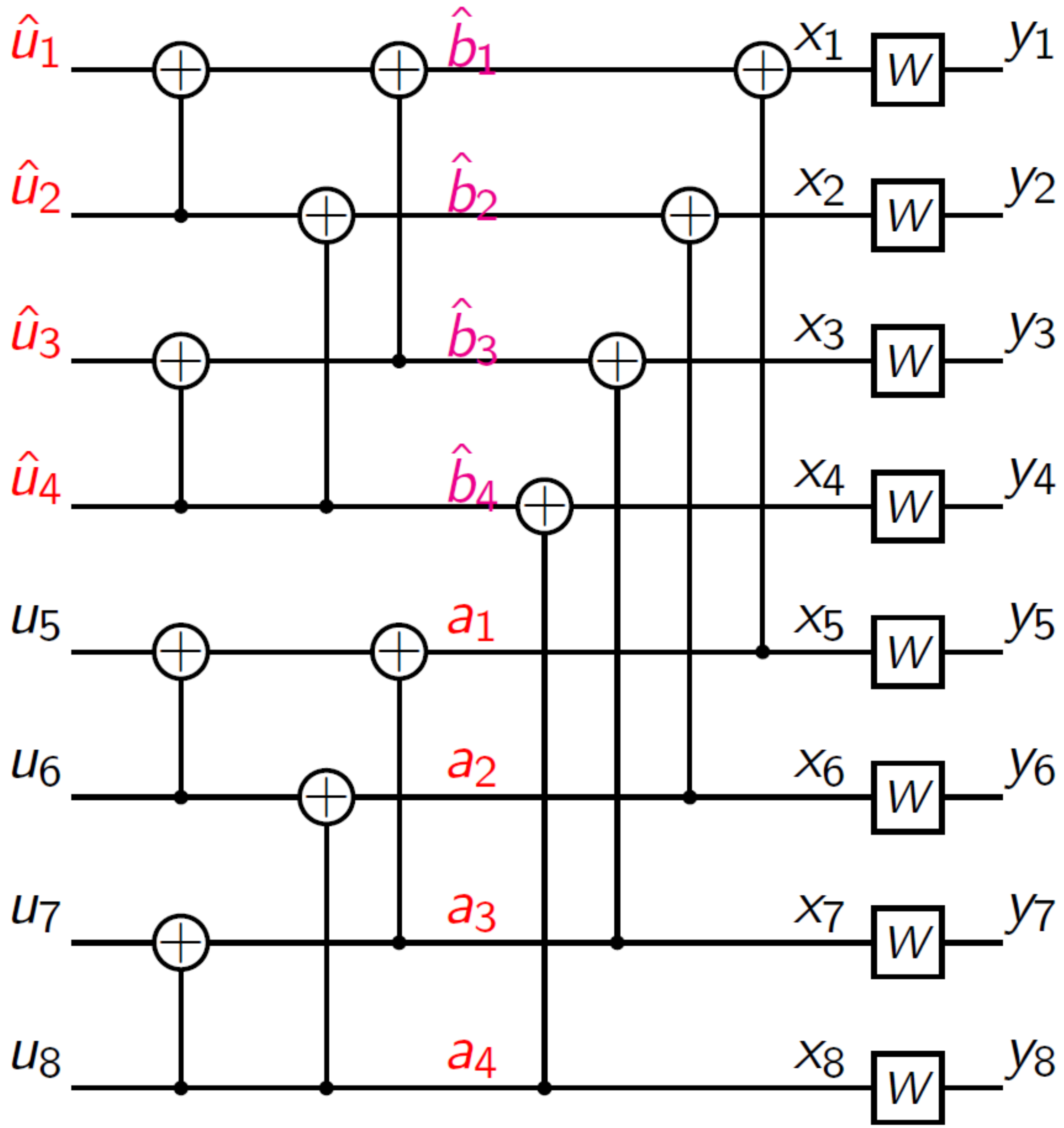
Successive Cancellation Decoder

First phase: treat **a** as noise, decode (u_1, u_2, u_3, u_4)



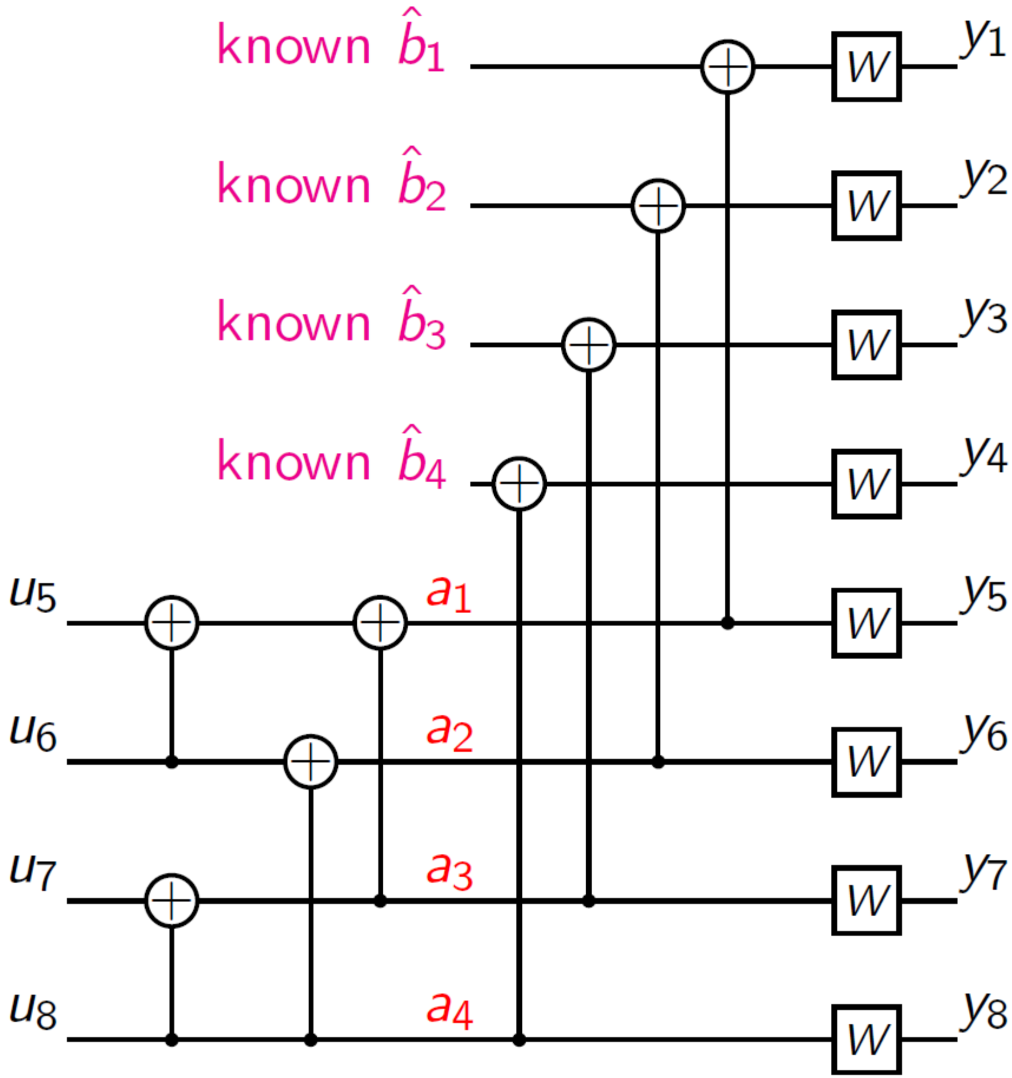
Successive Cancellation Decoder

End of first phase



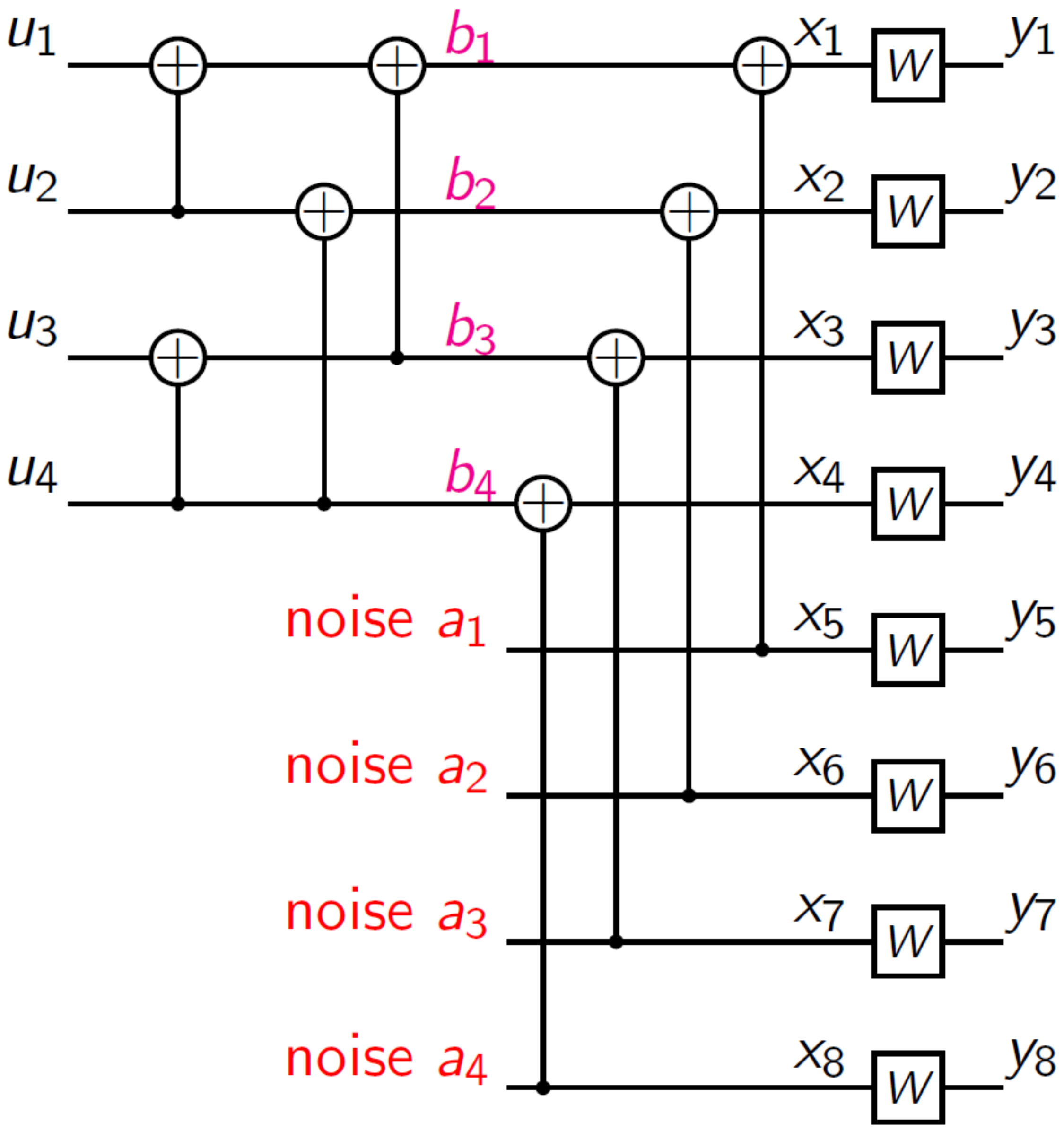
Successive Cancellation Decoder

Second phase: Treat $\hat{\mathbf{b}}$ as known, decode (u_5, u_6, u_7, u_8)



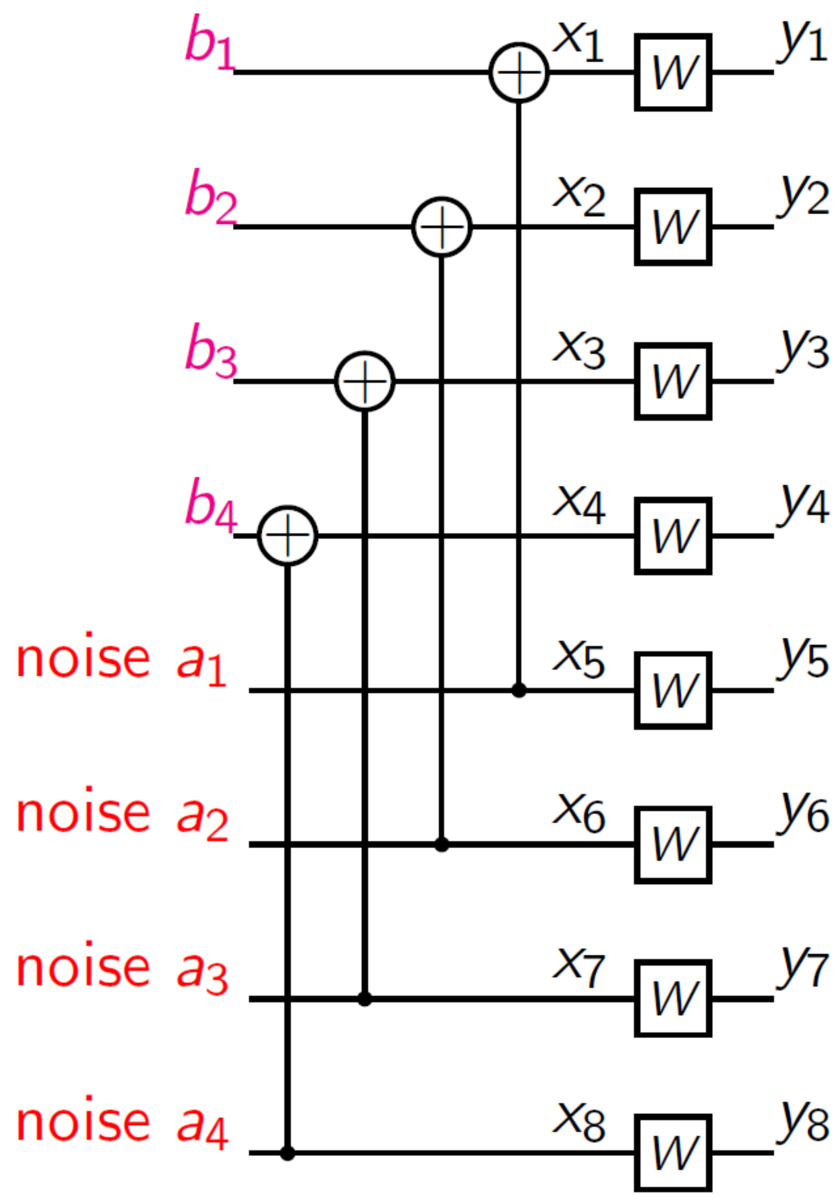
Successive Cancellation Decoder

First phase in detail



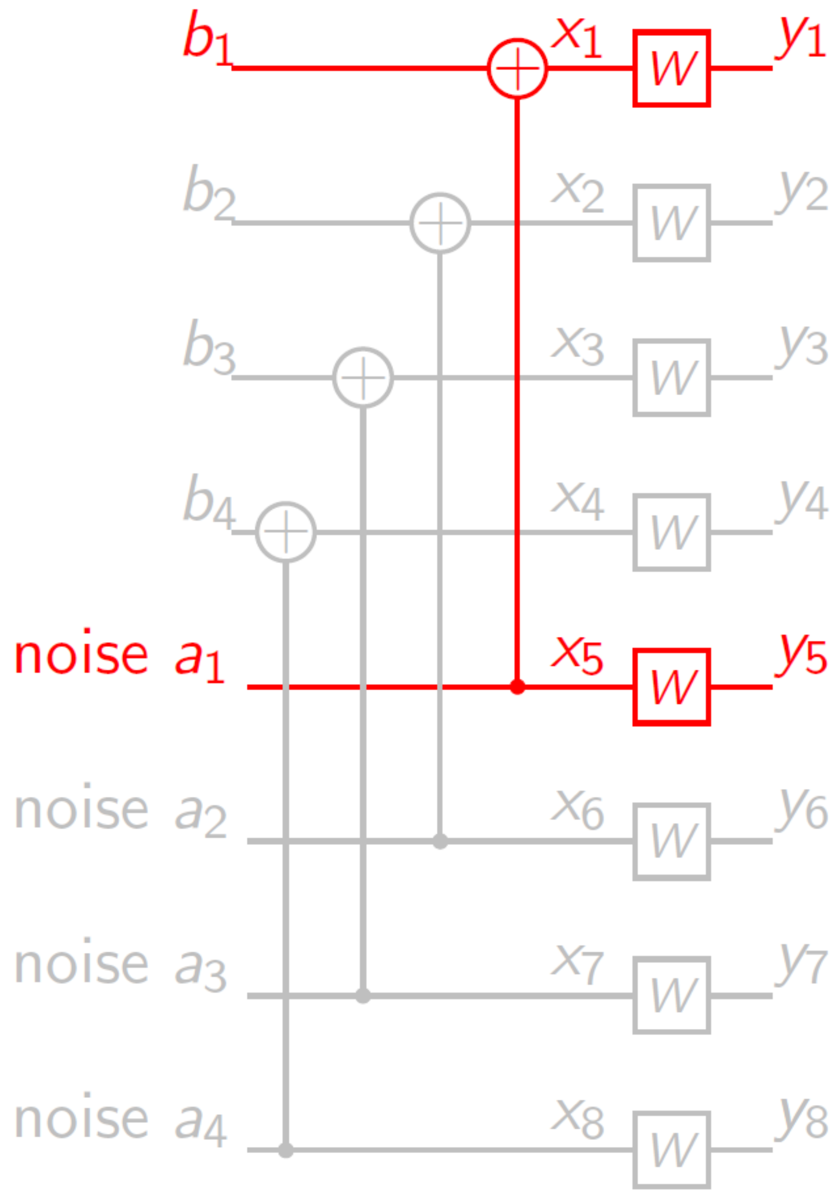
Successive Cancellation Decoder

Equivalent channel model



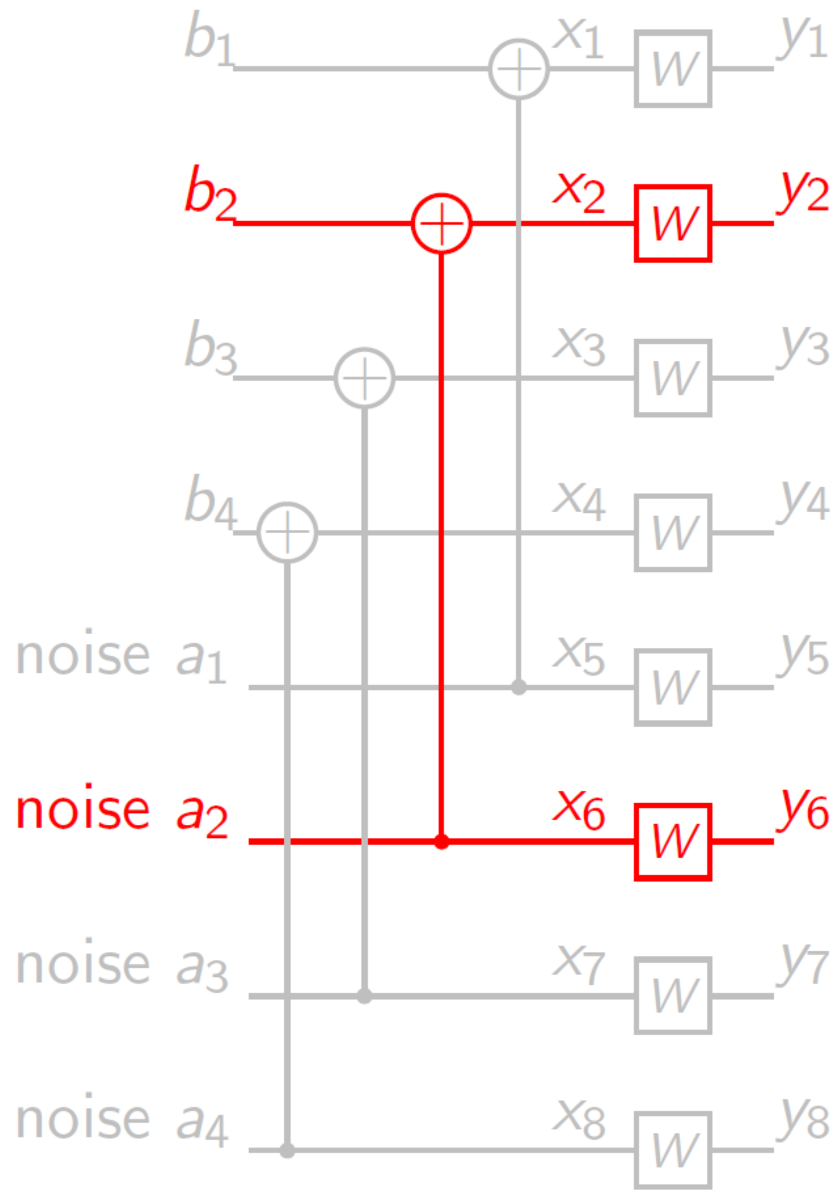
Successive Cancellation Decoder

First copy of W^-



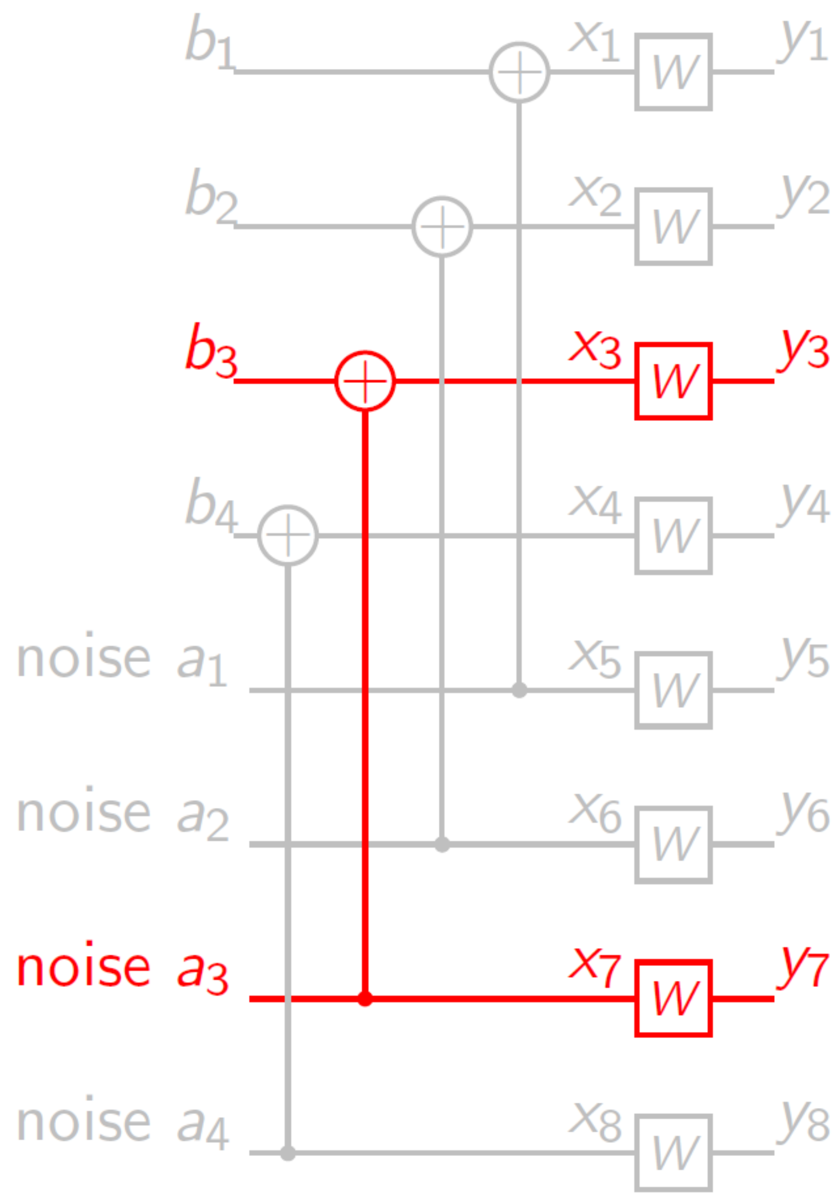
Successive Cancellation Decoder

Second copy of W^-



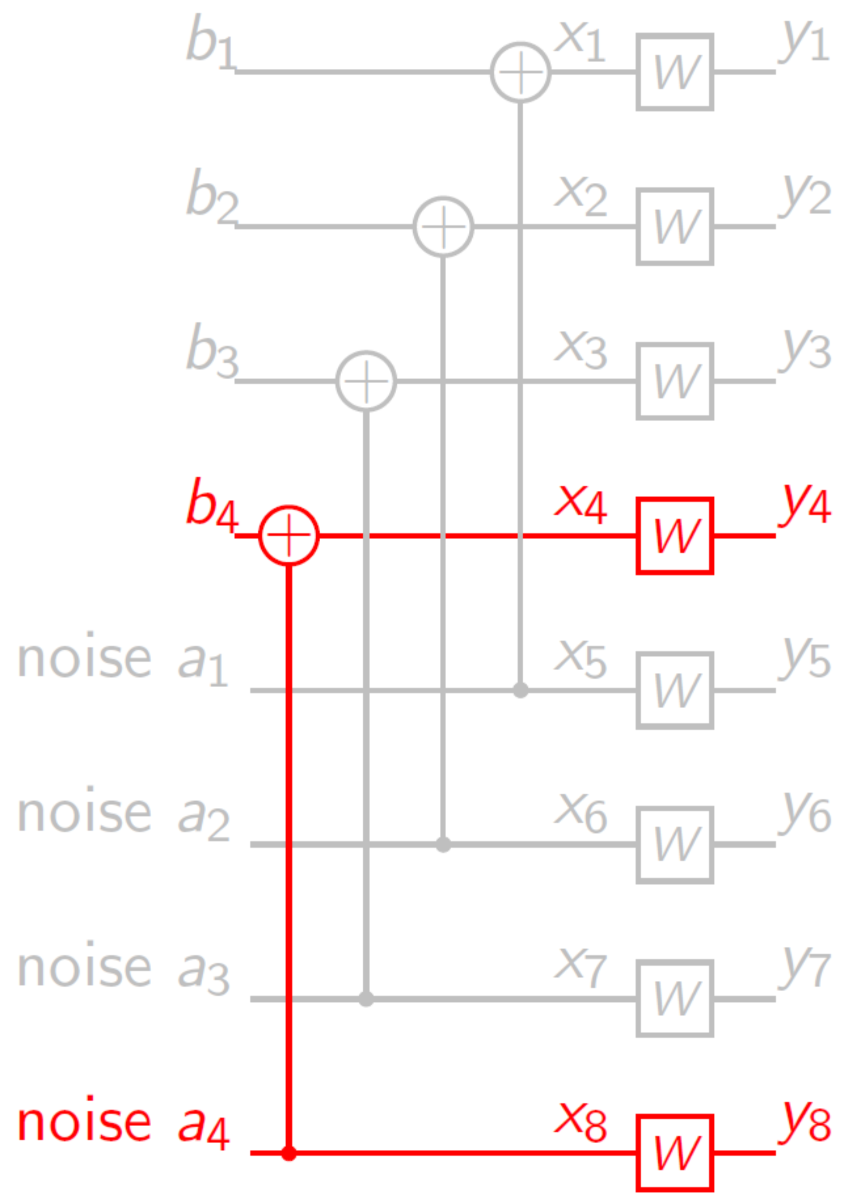
Successive Cancellation Decoder

Third copy of W^-

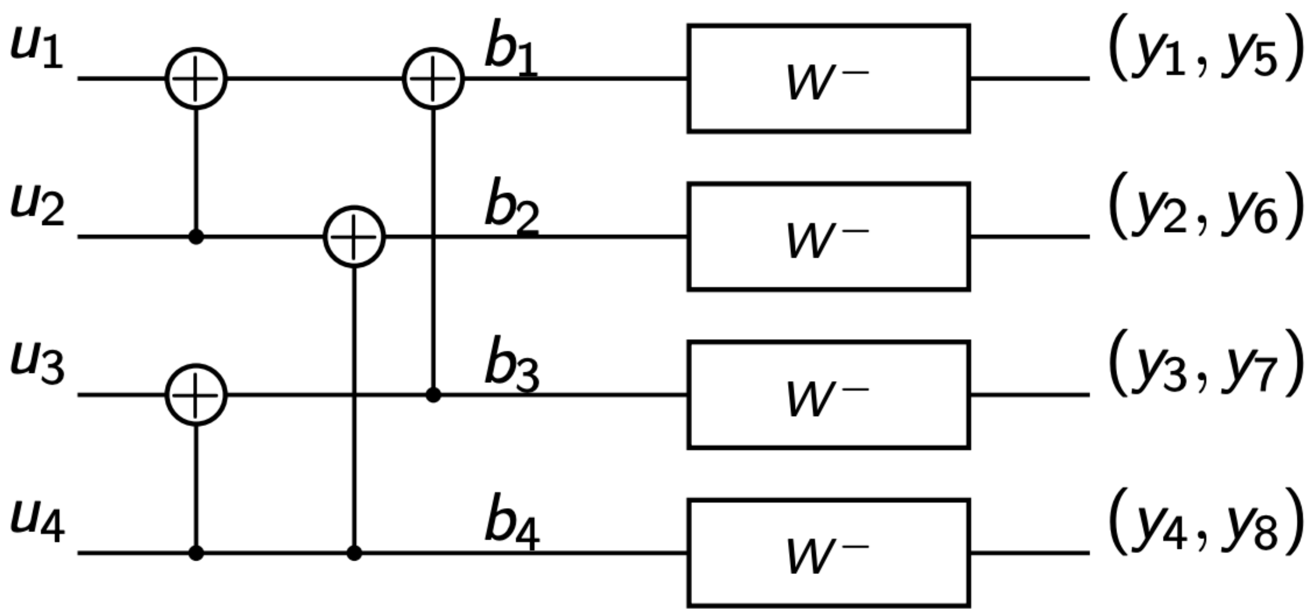


Successive Cancellation Decoder

Fourth copy of W^-

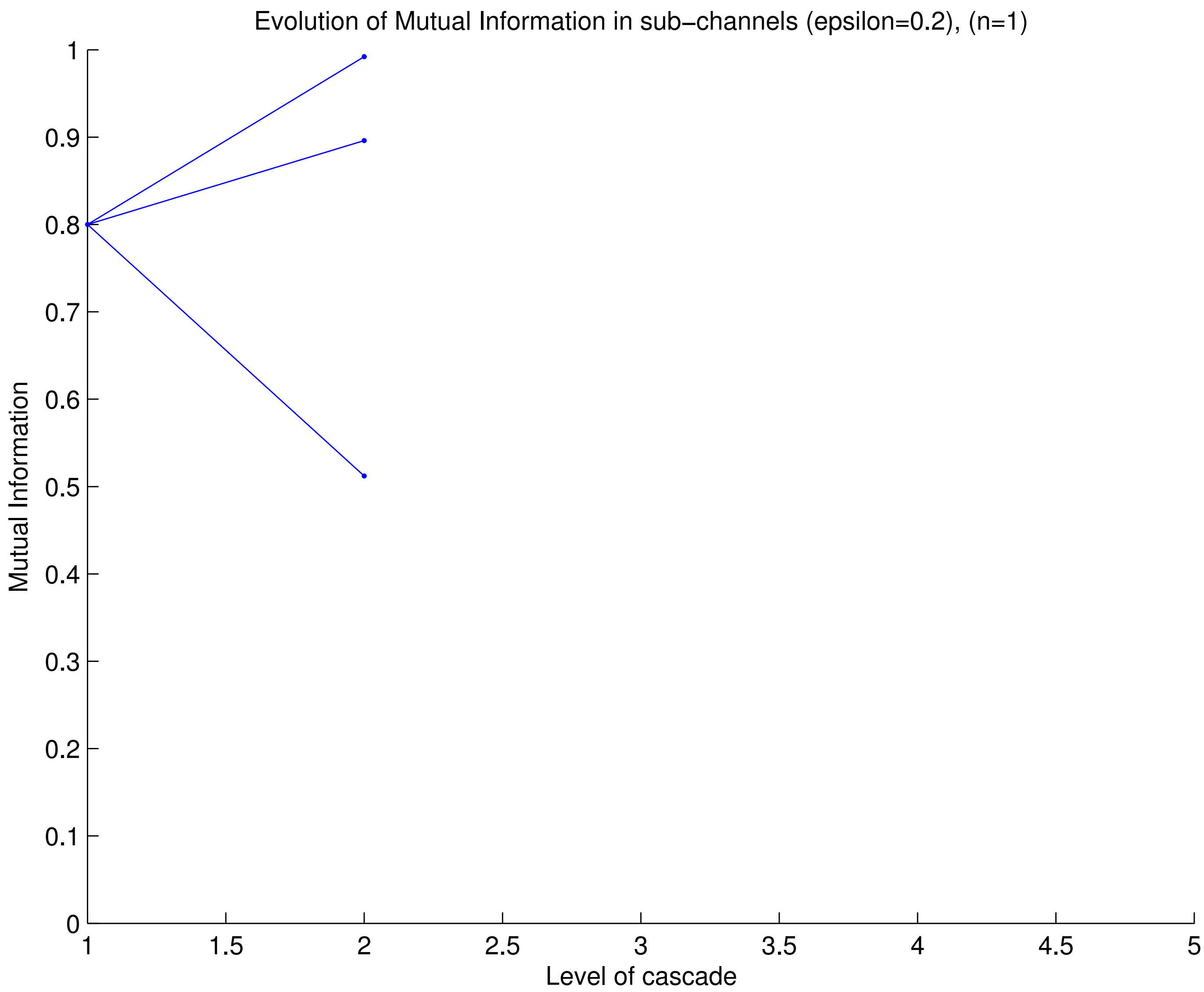


Reduction to 4×4



- Decoding the lower block u_5, u_6, u_7, u_8 is done similarly with a 4×4 block of W^+ channels

Mutual Information in TEC



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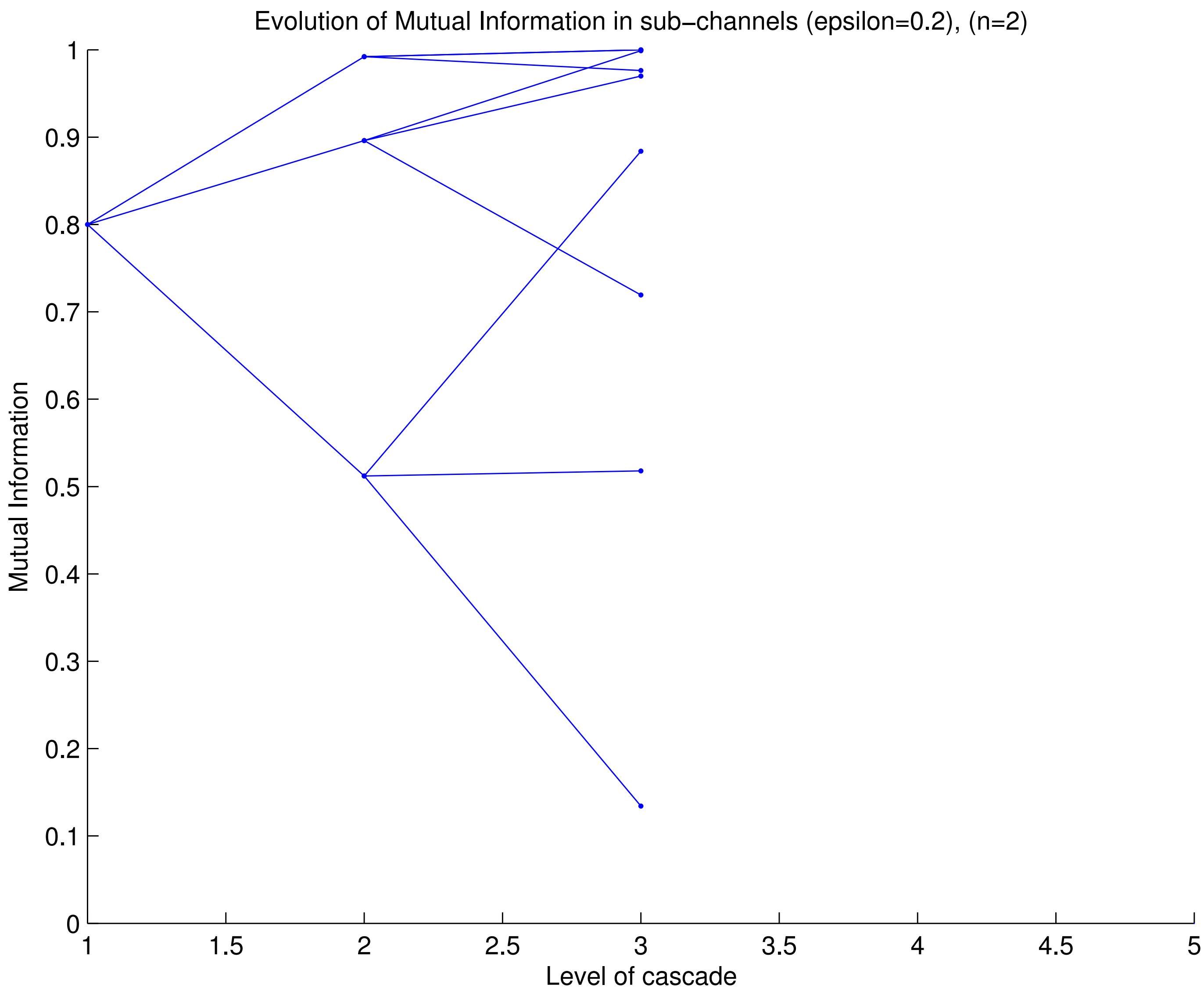

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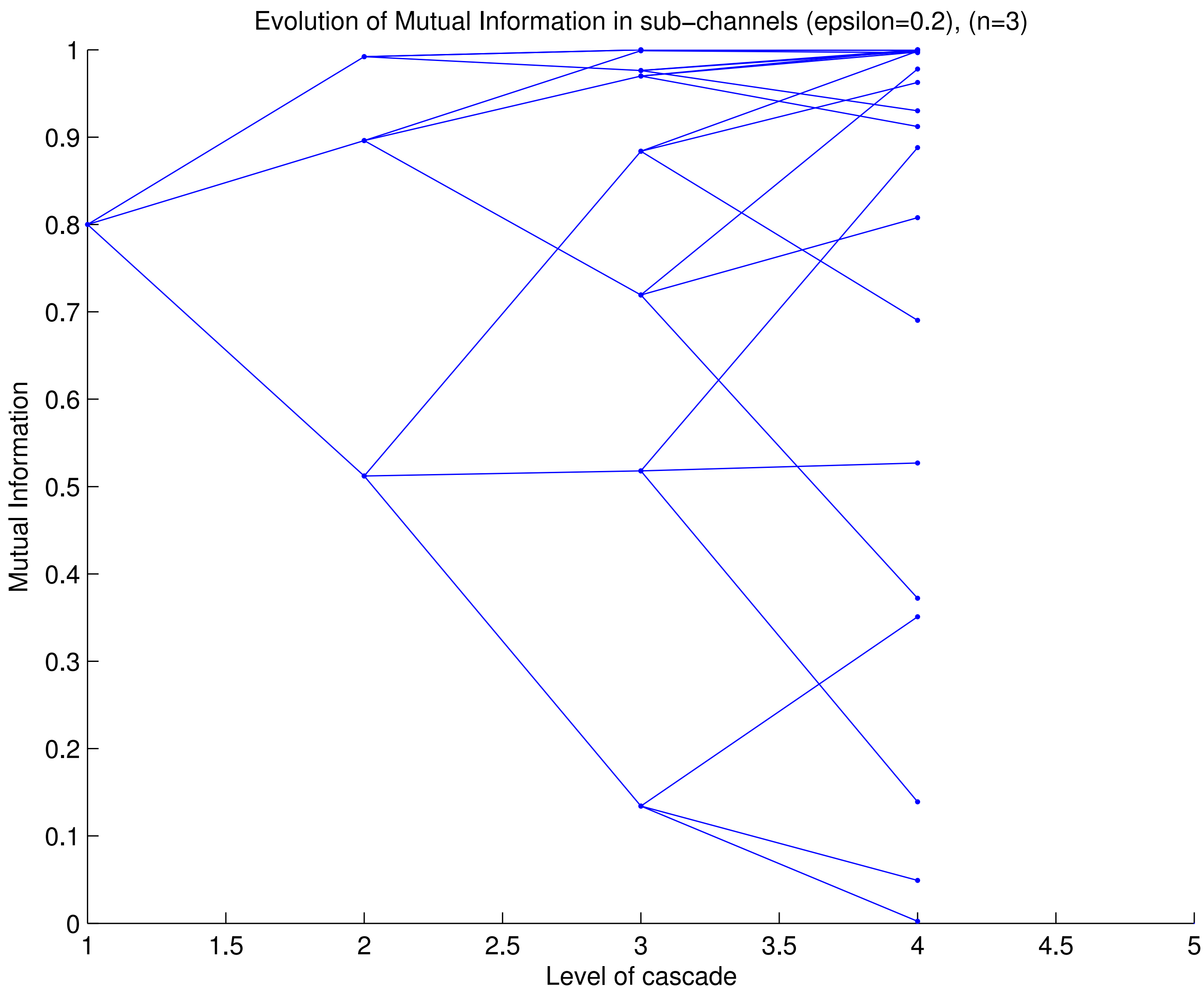
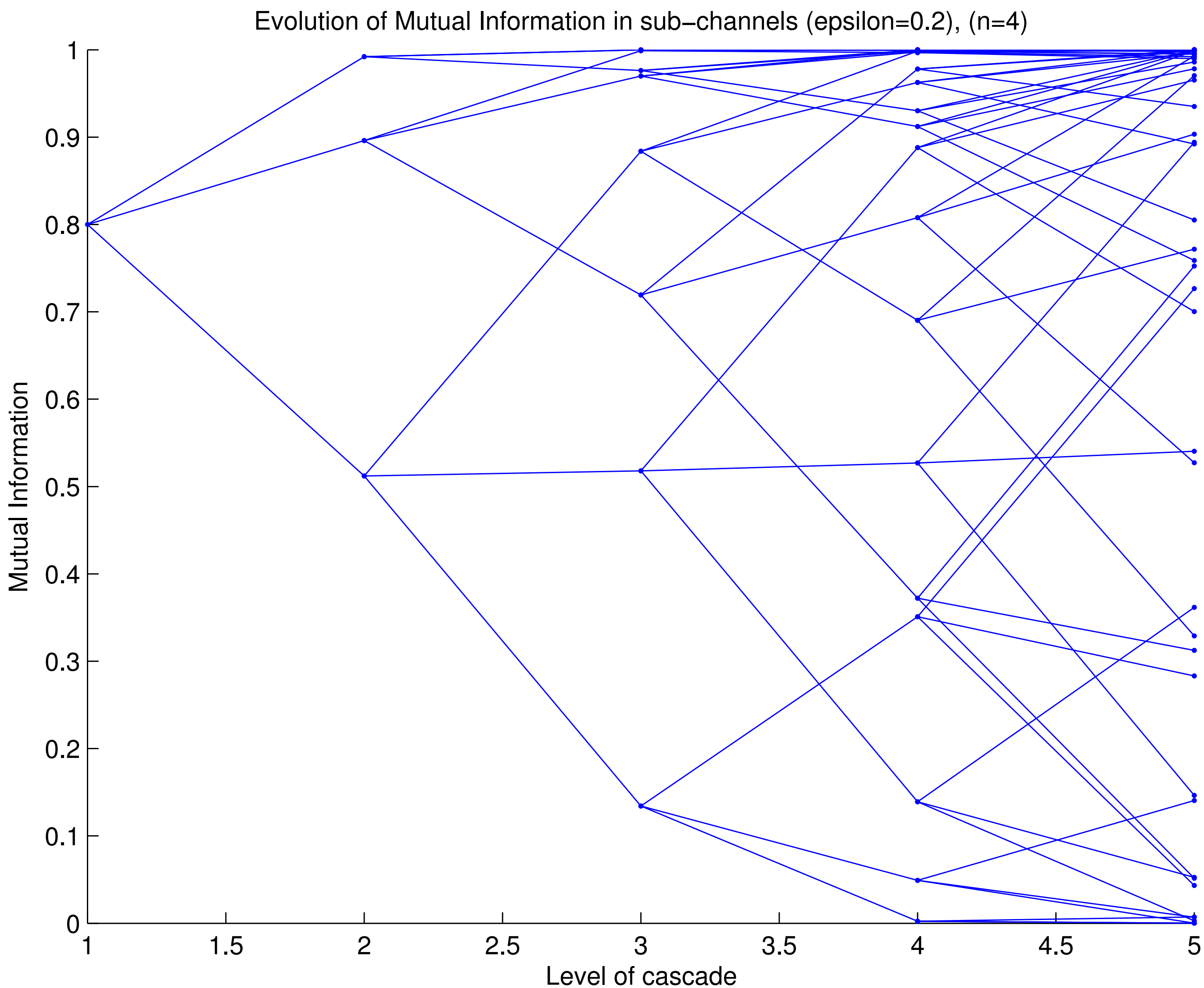
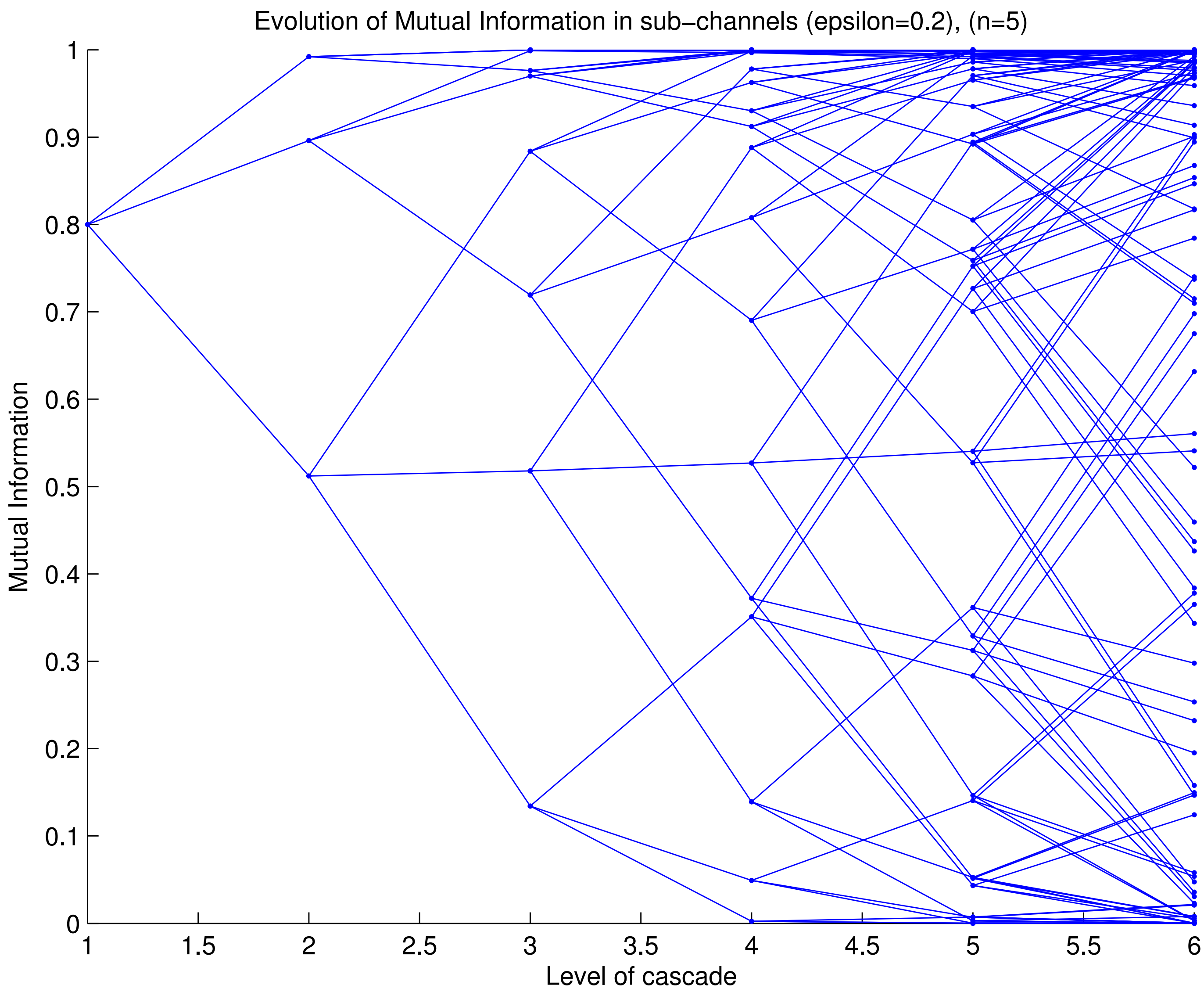


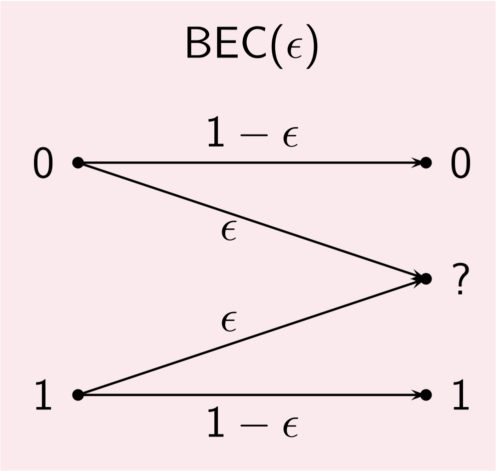
Mutual Information in TEC



Mutual Information in TEC



Capacity of the binary erasure channel (BEC)



$$
\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= H(X) - H(X)\epsilon - 0\,P(Y=0) - 0\,P(Y=1) \\
&= (1-\epsilon)H(X)
\end{aligned}
$$

Picking $X \sim Ber(\frac{1}{2})$, we have $H(X) = 1$. Thus, the capacity of BEC is $C = 1 - \epsilon$ Capacity of the BEC with erasure probability ϵ is $C = 1 - \epsilon$

References

- Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels, E. Arikan - IEEE Transactions on information Theory, 2009
- A Short Course on Polar Coding, E. Arikan, 2016