Final Examination Problems

1. Small questions (15 points).

![Auto-correlation function](image)

Figure 1: Auto-correlation function

a. (4 points) The autocorrelation function of a WSS process $X(t)$ for $0 \leq t \leq 4$ is given in Fig. 1. What is the variance of $X(-2) - X(7)$?

b. (4 points) Assume we are able to observe the process $X(t)$ with autocorrelation function in Fig. 1 for $5 \leq t \leq 10$ and we would like to estimate $X(17)$. What is the MMSE estimate and what is the corresponding MSE?

c. (4 points) Assume $X(t) = A \cos(\omega t) + B \sin(\omega t)$, where $A$ and $B$ are both $N(0,1)$ and independent of each other and $\omega$ is constant. Is the process SSS? Justify your answer.

d. (3 points) Assume $X(t) = A \cos(\omega t) + B \sin(\omega t)$ but nothing is known about the joint distribution of $A$ and $B$ and $\omega$ is constant. What can you say about the MS continuity of the process $X(t)$. Justify your answer.

2. Bernoulli i.i.d. Process (27 points).

Packets arriving at a server can be modeled as an i.i.d. Bernoulli Process with probability of arrival at each time $n = 1, \ldots$ equal to $p$. Let $S_n$ be the total number of packets in the server at time $n$.

a. (4 points) Is $S_n$ Markov? Is it an independent increment process? Is it stationary (SSS or WSS)? Justify your answer.

b. (4 points) Specify the first and second order pmf of $S_n$.

c. (4 points) Assume $p = 0.02$. Given that there are 5 packets in the server at time $n = 100$, what is the probability that 3 packets arrived in the interval $20 < n \leq 50$? Give an approximate numerical value for this probability by using the Poisson approximation for a Binomial r.v.
d. (4 points) Assume $p = 0.02$ and that the server queue has total size 100. Approximate
the probability that the server becomes full at $n \leq 1000$ by using the central limit
theorem. You can keep your answer in terms of the Q function.

e. (4 points) Let $T_1, T_2, \ldots$ be a discrete time process denoting the interarrival times,
  i.e., $T_1$ is the time until the first packet arrival, $T_2$ is the time between the first and
  the second packet arrivals. Specify the process

f. (7 points) It has been observed that after a rainy day the number of days until it rains
again is a geometrically distributed random variable with parameter 0.1 independent
of the past. What is the probability that it rains both on the 5th and 13th of
December? (Without loss of generality, you can assume that the process starts in the
beginning of the year and time to the first rainy day is also geometrically distributed
with parameter 0.1.)

3. **Convergence (28 points).**

Consider a sequence of random variables $\{Y_n, n \geq 1\}$. For each integer $j$ and corresponding
interval $[5j, 5j+1]$, $Y_n = 1$ at a single randomly chosen $n \in [5j, 5j+1]$. And, $Y_n = 0$ for
the remaining $n \in [5j, 5j+1]$.

a. (4 points) Does $Y_n$ converge in probability? Justify your answer.

b. (4 points) Does $Y_n$ converge with probability 1? Justify your answer.

c. (4 points) Does $Y_n$ converge in mean square? How about in distribution? Justify
  your answer.

Let the sample space be $S = [0, 1]$ with a uniform probability distribution. Consider the
sequence of random variables $X_n(s) = s + s^n$.

d. (5 points) Does the sequence $X_n$ converge with probability one? If so, to what limit?
  Justify your answer.

Let $\{Y_i, i \geq 1\}$ be a sequence of iid random variables such that $P\{Y_i = 1\} = \delta$ and
$P\{Y_i = 0\} = 1 - \delta$. Let $m$ be an arbitrary fixed integer, and define $S_n = X_1 + \cdots + X_n$.
Evaluate the following limits:

e. (6 points) $\lim_{n \to \infty} \sum_{i: n\delta - m \leq i \leq n\delta + m} P(S_n = i)$.

f. (5 points) $\lim_{n \to \infty} \sum_{i: (n\delta - \frac{1}{m}) \leq i \leq n(\delta + \frac{1}{m})} P(S_n = i)$.

4. **Continuous-time Process.** (30 points)

Consider a particle moving at a constant velocity along a straight line and suppose that
collisions involving this particle occur at a Poisson rate $\lambda$, i.e. the number of collisions
upto time $t$ can be represented by a Poisson process $N(t)$ with rate $\lambda$. Each time the
particle suffers a collision it reverses its direction without changing the magnitude of its
velocity, i.e if it is velocity were $X_0$, it becomes $-X_0$; if it were $-X_0$, it becomes $X_0$.
Its initial velocity $X_0$ at time $t = 0$ is either $+1$ or $-1$, where the sign represents the
direction, such that $P(X_0 = 1) = P(X_0 = -1) = 1/2$, independent of everything else.

a. (3 points) Let $X(t)$ be the velocity of the particle at time $t$. Express $X(t)$ in terms
  of $X_0$ and $N(t)$.
b. (3 points) Let $D(t)$ be the position of the particle at time $t$. Express $D(t)$ in terms of $X(t)$. Assume that the particle is at position 0 at time $t = 0$.

c. (6 points) Find the mean and the autocorrelation function of $X(t)$. Hint: note that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

d. (5 points) Find the mean and the variance of $D(t)$.

e. (7 points) Is $X(t)$ WSS? Is it SSS? Justify your answer.


g. (4 points) Is $X(t)$ mean ergodic? Justify your answer.