Sample Midterm Problems

The following are old midterm problems. The midterm will cover the material in Lecture notes 1, 2, 3, and 4 up to page 4-11, and homeworks 1–4 (including extra problems).

1. **Inequalities.** Label each of the following statements with =, ≤, ≥, or NONE. Label a statement with = if equality always holds. Label a statement with ≥ or ≤ if the corresponding inequality holds in general and strict inequality holds sometimes. If no such equality or inequality holds in general, label the statement as NONE. Justify your answers.
   a. $E(X_1X_2|X_3)$ vs. $E(X_1|X_3)E(X_2|X_3)$ if $X_1$ and $X_2$ are independent.
   b. $E[Var(X|Y,Z)]$ vs. $E[Var(X|Y)]$.
   c. $E[Var(X|Y)]$ vs. $E[Var(X|g(Y))]$. (Hint: Use the result of part (b).)
   d. $E_Z [E(X^2|Z) E(Y^2|Z)]$ vs. $[E_Z(Cov(X,Y|Z))]^2$.
   e. $E \left( \log_2 \left( 1 + \sqrt{X} \right) \right)$ vs. 1 if $X \geq 0$ and $E(X) \leq 1$.
   f. $P\{X^2 > 16\}$ vs. 1/8 if $E(X^4) = E(Y^4) = 2$.

2. **Random interval length.** Let random variable $X \sim U[0,1]$. Then $X$ divides the unit interval into two subintervals $[0,X]$ and $(X,1]$ and one of these subintervals is picked in the following way. Let the random variable $Y \sim U[0,1]$ be independent of $X$ and pick the subinterval that $Y$ falls into. Define the length of the subinterval chosen as $L$. Thus $L = X$ if $Y \leq X$ and $L = 1 - X$ if $Y > X$.
   a. Find the probability density function of $L$.
   b. Suppose that you do not know $X$ or $L$ but you observe $Y$. What is the minimum MSE estimate of $L$ given $Y$? Hint: this can be solved without finding any pdfs. You need only note that $L$ is a function of $X$ and $Y$.

3. **Wireless Channel.**

Consider the wireless communication channel with received signal

$$Y = H^T X + Z,$$

where $X^T = [X_1 X_2]$ is the transmitted signal, $H^T = [H_1 H_2]$ is the channel gain vector, and $Z$ is the noise. Assume that $E(X) = 0$, $Var(X_1) = Var(X_2) = P$, $\rho_{X_1,X_2} = \rho$, $E(H_1) = E(H_2) = 1$, $Var(H_1) = Var(H_2) = 1$, $\rho_{H_1,H_2} = 0$, $E(Z) = 0$, $Var(Z) = N$, and $X, H_1, H_2, Z$ are independent.

Find the best linear MSE estimate of $U = (X_1 + X_2)$ given $Y$. Your answer should be in terms only of $P, N,$ and $\rho$. 
4. **Gaussian Random Vector.**

Let \([X \ Y \ Z]^T\) be a zero mean Gaussian random vector with covariance matrix

\[
\Sigma = \begin{bmatrix}
    a & b_1 & 0 \\
    b_1 & a & b_2 \\
    0 & b_2 & a
\end{bmatrix}.
\]

where the \(a\) and \(b\) parameters are real-valued NON-ZERO constants.

a. Label each of the following statements as TRUE, FALSE, or NEITHER. Label a statement as TRUE if it always holds. Label a statement as FALSE if it never holds. Label a statement as NEITHER if it may or may not hold. Justify your answers.

i. \(a > 0\).

ii. \(b_1 > 0\).

iii. \(a^2 - b_1^2 - b_2^2 > 0\).

iv. \(X\) and \(Z\) are independent.

v. \(X\) and \(Z\) are conditionally independent given \(Y\).

b. Let \(\hat{X}\) be the best MSE estimate of \(X\) given \(Y\). Specify the joint pdf of \(\hat{X}\) and \(Z\). Your answer should be in terms only of the \(a\) and \(b\) parameters.

5. **Two coins.**

You are given two coins, coin 1 has bias \(1/2\) and coin 2 has a randomly selected bias \(P \sim U[0,1]\). You pick one of them at random and flip it twice. Observing the outcomes of these two coin flips, you wish to decide which coin was selected. Hence, let \(\Theta = 1\) if coin 1 is selected and \(\Theta = 2\) if coin 2 is selected with \(p_\Theta(1) = p_\Theta(2) = 1/2\). Let \(X_i = 1\) if the outcome of flip \(i\) is heads and \(X_i = 0\) if the outcome is tails for \(i = 1, 2\). Assume that \(X_1\) and \(X_2\) are conditionally independent given the value of the bias of the selected coin.

a. Find the estimate \(\hat{\Theta}(X_1, X_2) \in \{1, 2\}\) that minimizes the probability of error \(P\{\hat{\Theta} \neq \Theta\}\). Your answer should be explicit in terms only of \(X_1\) and \(X_2\).

b. Find the minimum probability of error.