1. **Inequalities.** Label each of the following statements with =, ≤, ≥ or NONE. Label a statement with = if equality always holds. Label a statement with ≥ or ≤ if strict inequality is possible. If no such equality or inequality holds in general, label the statement as NONE. Justify your answers.

   a. \( P(A) \) vs. \( 1 - (P(A^c, B) + P(A^c, B^c)) \).

   b. \( P(A | B) \) vs. \( P(A | B, C) \).

   c. \( P(A | B^c) \) vs. \( P(A) \) if \( P(A | B) \geq P(A) \).

   d. \( P(\bigcap_{i=1}^n A_i) \) vs. \( 1 - \sum_{i=1}^n P(A_i^c) \).

   **Solution**

   a. =. Using law of total probability, we obtain
   
   \[
P(A) = 1 - P(A^c) = 1 - (P(A^c, B) + P(A^c, B^c)).
   \]

   b. NONE. By definition
   
   \[
P(A|B, C) = \frac{P(A, B, C)}{P(B, C)} \quad P(A|B) = \frac{P(A, B)}{P(B)}.
   \]
   
   Now the numerator and denominator in the first expression are smaller than the corresponding numerator and denominator in the second. Therefore, no general inequality relation holds.

   c. ≤. By the law of total probability,
   
   \[
P(A) = P(A | B)P(B) + P(A | B^c)P(B^c).
   \]
   
   Therefore
   
   \[
P(A | B^c)P(B^c) = P(A) - P(A | B)P(B).
   \]

   We can bound \( P(A | B^c) \) as follows:
   
   \[
P(A | B^c) \leq \frac{P(A) - P(A | B)P(B)}{P(B^c)} \quad \text{since} \quad P(A | B) \geq P(A)
   \]
   
   \[
   \leq \frac{P(A) - P(A)P(B)}{P(B^c)} = \frac{P(A)(1 - P(B))}{1 - P(B)} = P(A).
   \]
d. By DeMorgan’s law,
\[ P(\cap_{i=1}^{n} A_i) = P((\cup_{i=1}^{n} A_i^c)^c) = 1 - P(\cup_{i=1}^{n} A_i^c). \]

Now, by the union of events bound
\[ P(\cup_{i=1}^{n} A_i^c) \leq \sum_{i=1}^{n} P(A_i^c). \]

Substituting in the previous equation gives
\[ P(\cap_{i=1}^{n} A_i) \geq 1 - \sum_{i=1}^{n} P(A_i^c). \]

2. **Mutual independence** Give an example of events \( A_1, A_2, A_3, \ldots, A_n \) such that \( P(\cap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i) \), but \( A_1, A_2, A_3, \ldots, A_n \) are not mutually independent. Any example with \( n \geq 3 \) is sufficient.

**Solution**

Consider a sample space \( \Omega = \{1, 2, 3, 4, 5, 6, 7, 8\} \), where each number has a probability \( \frac{1}{8} \). Let events \( A_1 = A_2 = \{1, 2, 3, 4\} \), and \( A_3 = \{1, 5, 6, 7\} \). One can see that \( P(A_1) = P(A_2) = P(A_3) = \frac{1}{2} \) and \( P(\cap_{i=1}^{3} A_i) = \frac{1}{8} \), so \( P(\cap_{i=1}^{3} A_i) = \prod_{i=1}^{3} P(A_i) \) but obviously \( A_1 \) and \( A_2 \) are not independent.

3. **Additive Gaussian Noise channel.** A communication channel has a real-valued input signal \( x \) and an output \( Y = x + 2Z \), where \( Z \sim N(0, 0.09) \). Suppose that \( x = -1 \) is sent. Use a Q-function table to find the probability of the event \( \{Y > 0\} \).

**Solution**

Since \( Z \sim N(0, 0.09) \), \( Y = -1 + 2Z \sim N(-1, 0.36) \). Now, we can express \( Y = 0.6W - 1 \), where \( W \sim N(0, 1) \). Thus
\[ P\{Y > 0\} = P\{0.6W - 1 > 0\} = P\{W > \frac{1}{0.6}\} \approx Q(1.67) \approx 0.04776. \]

4. **Exponential random variable.** Let \( X \sim \text{Exp}(0.1) \) be the distribution of the time it takes to serve one customer at a bank (in minutes).

a. The teller has just started to serve the person ahead of you. What is the probability that you have to wait 10 minutes or more?

b. The person ahead of you has now been served for 10 minutes. What is the probability that you have to wait another 10 minutes or more?

c. For a general exponential r.v. \( X \sim \text{Exp}(\lambda) \), compute \( P\{X > a + b \mid X > a\} \) for \( a, b \geq 0 \) and interpret the result.

**Solution**
a. It is easy to see that the CDF of $X \sim \text{Exp}(\lambda)$ is $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. For $\lambda = 0.1$, we have $\Pr\{X > 10\} = 1 - F_X(10) = e^{-1}$.

b. We want $\Pr\{X > 20 \mid X > 10\}$. By definition,
$$
\Pr\{X > 20 \mid X > 10\} = \frac{\Pr\{X > 20, X > 10\}}{\Pr\{X > 10\}} = \frac{\Pr\{X > 20\}}{\Pr\{X > 10\}} = \frac{e^{-2}}{e^{-1}} = e^{-1}
$$

c. Similar as before,
$$
\Pr\{X > a + b \mid X > a\} = \frac{\Pr\{X > a + b, X > a\}}{\Pr\{X > a\}} = \frac{\Pr\{X > a + b\}}{\Pr\{X > a\}} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b}
$$

The result says that the probability of waiting an additional $b$ minutes is independent of the wait time $a$ that has already passed. Therefore, the exponential distribution is said to be memoryless.

5. Lognormal pdf. Let $X \sim \mathcal{N}(0, \sigma^2)$. Find the pdf of $Y = e^X$ (known as the lognormal pdf).

**Solution**

If $Y = e^X$ then $\Pr(Y \leq y) = 0$ for $y \leq 0$. For $y > 0$,
$$
\Pr(Y \leq y) = \Pr(e^X \leq y) = \Pr(X \leq \ln y) = F_X(\ln y) \Rightarrow F_Y(y) = \begin{cases} 0 & y \leq 0 \\ F_X(\ln y) & y > 0 \end{cases}
$$

Obviously $f_Y(y) = 0$ for $y < 0$. For $y > 0$,
$$
f_Y(y) = \frac{d}{dy} F_Y(y) = F'_X(\ln y) \frac{d}{dy} \ln y = \frac{1}{y} f_X(\ln y).
$$

Applying this result to $X \sim \mathcal{N}(0, \sigma^2)$, we obtain
$$
f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{e^{-(\ln y)^2/2\sigma^2}}{y\sqrt{2\pi}\sigma} & y > 0 \end{cases}
$$

In this case $Y$ is said to have a lognormal pdf.

6. Nonlinear processing. Let $X \sim \mathcal{U}[-1, 1]$. Define the random variable
$$
Y = \begin{cases} X^2 + 1, & \text{if } |X| \geq 0.5 \\ 0, & \text{otherwise} \end{cases}
$$
Find and sketch the cdf of $Y$, $F_Y(y)$.

**Solution**

For $Y < 0$, $F_Y(y) = 0$.

For $Y = 0$, $F_Y(y) = \Pr\{|X| < 0.5\} = 0.5$.

For $0 < Y < 1.25$, $F_Y(y) = 0.5$.

For $1.25 \leq Y \leq 2$,

$$F_Y(y) = 0.5 + \Pr\{-\sqrt{y-1} \leq X \leq -0.5 \text{ or } 0.5 \leq X \leq \sqrt{y-1}\}$$

$$= 0.5 + 2\Pr\{0.5 \leq X \leq \sqrt{y-1}\}$$

$$= 0.5 + (\sqrt{y-1} - 0.5) = \sqrt{y-1}.$$

Finally, for $Y > 2$, $F_Y(y) = 1$. The cdf of $Y$ is sketched in Figure 1.

![Figure 1: CDF of $Y$.](image)