Review Session #2 Solutions

1. *Random phase signal.* Let \( Y(t) = \sin(\omega t + \Theta) \) be a sinusoidal signal with random phase \( \Theta \sim U[-\pi, \pi] \). Find the pdf of the random variable \( Y(t) \) for fixed values of time \( t \) and radial frequency \( \omega \). Comment on the dependence of the pdf of \( Y(t) \) on time \( t \).

**Solution**

The general formula for calculating the pdf of a differentiable function of a continuous random variable is given in the lecture notes:

\[
    f_Y(y) = \sum_{\theta_k: f_{\Theta}(\theta_k) = y} \frac{f_{\Theta}(\theta)}{|dy/d\theta|_{\theta=\theta_k}}.
\]

We can apply that general formula to this special case. Here \( f_{\Theta}(\theta) = \frac{1}{2\pi}, \ -\pi \leq \theta \leq \pi \).

\[
    dy \over d\theta = \cos(\omega t + \theta).
\]

For \( y \in [-1, 1] \), there are two solutions to the equation \( y = \sin(\omega t + \theta) \) with \( \theta \in [-\pi, \pi] \). (Actually, there are two points where there is only one solution (\( y = \pm 1 \)) and one point where there are three (\( y = 0 \). We ignore these points since their probability is zero.) When \( y \) is plotted as a function of \( \theta \), it is apparent that the slopes at the two solutions do not depend on the phase shift \( \omega t \) (also \( f_{\Theta}(\theta) \) is uniform). Thus without loss of generality, we can take \( \omega t = 0 \), which gives the two solutions: \( \theta_1 = \arcsin y \) and \( \theta_2 = \pi - \theta_1 \). Thus,

\[
    \left| \frac{dy}{d\theta} \right|_{\theta_1} = \left| \frac{dy}{d\theta} \right|_{\theta_2} = \cos(\arcsin y) = \sqrt{1 - y^2}.
\]

Using the general formula for \( f_Y(y) \), we get

\[
    f_Y(y) = \frac{1}{2\pi} \left( \frac{1}{\sqrt{1-y^2}} + \frac{1}{\sqrt{1-y^2}} \right) = \begin{cases} \frac{1}{\pi \sqrt{1 - y^2}} & |y| < 1 \\ 0 & |y| > 1 \end{cases}
\]

Note that \( f_Y(y) \) does not depend on time \( t \), that is, \( Y(t) \) is time invariant (or stationary). (More on this later in the course.)

2. *Joint cdf or not.* Consider the function

\[
    G(x, y) = \begin{cases} 1 & \text{if } x + y \geq 0 \\ 0 & \text{otherwise.} \end{cases}
\]

Can \( G \) be a joint cdf for a pair of random variables? Justify your answer.

**Solution**

No. Note that for every \( x \),
\[
\lim_{y \to \infty} G(x, y) = 1.
\]

But for any genuine marginal cdf,

\[
\lim_{x \to -\infty} F_X(x) = 0 \neq 1.
\]

Therefore \(G(x, y)\) is not a cdf. Alternatively, assume that \(G(x, y)\) is a joint cdf for \(X\) and \(Y\), then

\[
P\{-1 < X \leq 2, -1 < Y \leq 2\} = G(2, 2) - G(-1, 2) - G(2, -1) + G(-1, -1)
\]

\[
= 1 - 1 - 1 + 0 = -1.
\]

But this violates the property that the probability of any event must be nonnegative.

3. \textit{Max to Min ratio}. Let \(X_1\) and \(X_2\) be two independent random variables, each uniformly distributed between 0 and 1, i.e., \(X_i \sim U[0, 1]\). Find and sketch the cdf of

\[
Y = \max(X_1, X_2) / \min(X_1, X_2).
\]

\textbf{Solution}

Since \(X_1 \sim U[0, 1]\) and \(X_2 \sim U[0, 1]\) are independent, their joint pdf is uniform over the square \(0 \leq x_1, x_2 \leq 1\). The cdf of \(Y\) can be found graphically by calculating the area of the region where \(Y \leq y\), as shown in Figure 1 on page 2.

![Diagram of the region for Y.](image)

Figure 1: Shaded region corresponds to \(Y = \max\{X_1, X_2\} / \min\{X_1, X_2\} \leq y\).
To see this more clearly, consider \( F_Y(y) \) for \( y \geq 1 \):

\[
F_Y(y) = P \left\{ \max \{X_1, X_2\} \leq y \right\} \\
= P \left\{ \min \{X_1, X_2\} \geq \frac{\max \{X_1, X_2\}}{y} \right\} \\
= P \left\{ X_2 \geq \frac{X_1}{y}, \ X_2 \leq X_1 \right\} + P \left\{ X_1 \geq \frac{X_2}{y}, \ X_1 < X_2 \right\} \\
= P \left\{ \frac{X_1}{y} \leq X_2 \leq X_1 \right\} + P \left\{ \frac{X_2}{y} \leq X_1 < X_2 \right\} \\
= 2P \left\{ \frac{X_1}{y} \leq X_2 \leq X_1 \right\} \quad \text{(by symmetry)} \\
= 2 \cdot \frac{1}{2} \cdot 1 \cdot \left( 1 - \frac{1}{y} \right) = \frac{y}{1-y}.
\]

Clearly \( F_Y(y) = 0 \) for \( y < 1 \). Note that \( f_Y(y) = 1/y^2 \) for \( y \geq 1 \).

4. **First available teller.** A bank has two tellers. The service times for tellers 1 and 2 are independent exponential random variables \( X_1 \sim \text{Exp}(\lambda_1) \) and \( X_2 \sim \text{Exp}(\lambda_2) \), respectively. You arrive at the bank and find that both tellers are busy but nobody else is waiting to be served. You are served by the first available teller once he/she is free. What is the probability that you are served by the teller 1?

**Solution**

The tellers’ service times are exponentially distributed, hence memoryless. Thus the service time distribution does not depend on my arrival time. The probability that I will be served by the first teller is

\[
P\{X_1 < X_2\} = \int_0^\infty \int_{x_1}^\infty \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} \, dx_2 \, dx_1 \\
= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2)x_1} \, dx_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\]

In other words, the probability of being served first by teller \( i \) is proportional to the teller’s service rate \( \lambda_i \).

5. **Correlation and independence.** Give a counterexample of this statement: “If random variables \( X \) and \( Y \) are uncorrelated then they are independent.”

**Solution**

Let \( X \sim U[-1, 1] \) and \( Y = X^2 \). One can see that \( E(XY) = E(X^3) = 0 = E(X)E(Y) \), which means that \( X \) and \( Y \) are uncorrelated. However, clearly \( X \) and \( Y \) are not independent because \( Y \) is a function of \( X \).

6. **Radar signal detection.** The received signal \( S \) for a radar channel is 0 if there is no target and a random variable \( X \sim \mathcal{N}(0, P) \) if there is a target. Both possibilities occur with equal
probability. Thus
\[ S = \begin{cases} 
0 & \text{with probability } \frac{1}{2} \\
X \sim \mathcal{N}(0, P) & \text{with probability } \frac{1}{2}.
\end{cases} \]

The radar receiver observes \( Y = S + Z \), where the noise \( Z \sim \mathcal{N}(0, N) \) is independent of \( S \). Find the optimal decoder for deciding whether \( S = 0 \) or \( S = X \) and its probability of error. Give your answer in terms of intervals of \( y \) and express the boundary points of the intervals in terms of \( P \) and \( N \).

Hint: You can cast this detection problem in the form discussed in class by defining the signal \( \Theta \) to be 0 if \( S = 0 \) and 1 if \( S = X \).

**Solution**

To cast this problem as a standard detection problem, we define a random variable \( \Theta \) by
\[ \Theta = \begin{cases} 
0 & \text{if } S = 0 \\
1 & \text{if } S = X
\end{cases} \]

Then \( p_\Theta(0) = p_\Theta(1) = \frac{1}{2} \). The optimal decoder \( \hat{\Theta}(\cdot) \) for \( \Theta \) uses the MAP rule: i.e., set \( \hat{\Theta}(y) = \theta \) where \( \theta \) maximizes the conditional pmf \( p_{\Theta|Y}(\theta|y) \). By Bayes rule,
\[ p_{\Theta|Y}(\theta|y) = \frac{f_{Y|\Theta}(y|\theta)p_\Theta(\theta)}{f_Y(y)} \Rightarrow \hat{\Theta}(y) = \begin{cases} 
1 & \text{if } \frac{f_{Y|\Theta}(y|1)}{f_{Y|\Theta}(y|0)} > 1 \\
0 & \text{otherwise}
\end{cases} \]

The likelihood ratio can be written
\[ \frac{f_{Y|\Theta}(y|1)}{f_{Y|\Theta}(y|0)} = \frac{1}{\sqrt{2\pi(P+N)}} e^{-\frac{y^2}{2(P+N)}} \cdot \frac{1}{\sqrt{2\pi N}} e^{-\frac{y^2}{2N}} = \sqrt{\frac{N}{P+N}} e^{y^2(\frac{P}{P+N})}, \]

hence
\[ \frac{f_{Y|\Theta}(y|1)}{f_{Y|\Theta}(y|0)} > 1 \Leftrightarrow \sqrt{\frac{N}{P+N}} e^{y^2(\frac{P}{P+N})} > 1 \Leftrightarrow y^2 > \frac{(P+N)N}{P} \ln \left( \frac{P+N}{N} \right). \]

Thus the MAP decision rule becomes
\[ \hat{\Theta}(y) = \begin{cases} 
0 & |y| \leq \sqrt{\frac{(P+N)N}{P} \ln \left( \frac{P+N}{N} \right)} \\
1 & \text{otherwise}
\end{cases} \]

To find the error probability, define \( \tau = \sqrt{\frac{(P+N)N}{P} \ln \left( \frac{P+N}{N} \right)} \). Then
\[ P_e = P\{\hat{\Theta}(Y) \neq \Theta\} \]
\[ = P\{\Theta(Y) = 1, \Theta = 0\} + P\{\hat{\Theta}(Y) = 0, \Theta = 1\} \]
\[ = P\{|Y| \geq \tau, \Theta = 0\} + P\{|Y| < \tau, \Theta = 1\} \]
\[ = p_\Theta(0) \left( \int_{-\infty}^{-\tau} f_{Y|\Theta}(y|0) \, dy + \int_{\tau}^{\infty} f_{Y|\Theta}(y|0) \, dy \right) + p_\Theta(1) \int_{-\tau}^{\tau} f_{Y|\Theta}(y|1) \, dy \]
\[ = \frac{1}{2} \left( 2 \int_{\tau}^{\infty} f_{Y|\Theta}(y|0) \, dy + \int_{-\tau}^{\tau} f_{Y|\Theta}(y|1) \, dy \right) \]
\[ = \frac{1}{2} \left( 2Q\left(\frac{\tau}{\sqrt{N}}\right) + \left( 1 - 2Q\left(\frac{\tau}{\sqrt{P+N}}\right) \right) \right) \]
\[ = Q\left( \sqrt{\frac{(P+N)}{P}} \ln \left( \frac{P+N}{N} \right) \right) + \frac{1}{2} - Q\left( \sqrt{\frac{N}{P}} \ln \left( \frac{P+N}{N} \right) \right). \]

In Figure 2 on page 5, the pdfs of noise and signal + noise intersect at ±\(\tau\). The decision region for "no signal" is the interval \([-\tau, +\tau]\). The error probability is the average of the probabilities of the tail of the noise and of the central region of signal + noise. In the example shown in Figure 2, SNR = 4 and \(P_e = 0.1780\).

Figure 2: PDFs of noise (\(N = 1\)) and radar signal + noise (\(P + N = 4\)).

7. **Mean-square inequality.** Let \(X\) and \(Y\) be random variables with finite means and variances. Show that
\[ P\{|X - Y| > \epsilon\} \leq \frac{E((X - Y)^2)}{\epsilon^2}. \]
Solution

Use the Markov inequality with $a = \frac{\epsilon^2}{E((X - Y)^2)}$:

$$P\{|X - Y| \geq \epsilon\} = P\{(X - Y)^2 \geq a \cdot E((X - Y)^2)\} \leq \frac{1}{a} = \frac{E((X - Y)^2)}{\epsilon^2}.$$