Review Session #3

1. **Independence vs. Conditional Independence** Give an example of random variables $X, Y,$ and $Z$ where $f_{X,Z}(x,z) = f_X(x)f_Z(z)$ but $f_{X,Z|Y}(x,z|y) \neq f_{X|Y}(x|y)f_{Z|Y}(z|y)$ i.e. independence does not imply conditional independence.

2. **Sum and difference.** Let $X$ and $Y$ be two random variables, and define $U = X - Y$ and $V = X + Y$. Find the minimum MSE linear estimate of $V$ given $U$ as a function of the random variables and $E(X)$, $E(Y)$, $\sigma_X$, $\sigma_Y$, $\rho_{X,Y}$, where $\sigma_X = \sqrt{\text{Var}(X)}$, $\rho_{X,Y} = \text{corr}(X,Y)$.

3. **Covariance matrices.** Which of the following matrices can be a covariance matrix? Justify your answer. Either construct a random vector $X$ with the given covariance matrix as a function of the i.i.d. zero mean unit variance random variables $Z_1, Z_2, Z_3$, or establish a contradiction as was done in lecture.

   (a) \[
   \begin{bmatrix}
   1 & 2 \\
   0 & 2 
   \end{bmatrix}
   \]

   (b) \[
   \begin{bmatrix}
   2 & 1 \\
   1 & 2 
   \end{bmatrix}
   \]

   (c) \[
   \begin{bmatrix}
   1 & 1 & 1 \\
   1 & 2 & 2 \\
   1 & 2 & 3 
   \end{bmatrix}
   \]

   (d) \[
   \begin{bmatrix}
   1 & 1 & 2 \\
   1 & 2 & 3 \\
   2 & 3 & 3 
   \end{bmatrix}
   \]

4. **Conditional Independence does not imply Independence.** In class, we saw an example in which two independent, identically distributed random variables conditioned on a third random variable were no longer independent. Here, we examine an example of the opposite case: is it possible for conditionally independent random variables to be not independent?

   Suppose $X_3 \sim U[0,1]$, and given $X_3$: $X_1, X_2 \overset{i.i.d.}{\sim} \text{Bern}(X_3)$. Show that $X_1, X_2$ are not independent, although they are conditionally independent given $X_3$ as given. Work out the joint distribution $P_{X_1, X_2}$.

   Hint: the Beta function is $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$.

5. **Additive-noise channel with path gain.** Consider the output $Y$ of an additive-noise channel with path gain, where $X$ and $Z$ are zero mean and uncorrelated, and $a$ and $b$ are constants. Find the MMSE linear estimate of $X$ given $Y$ and its MSE in terms only of $\sigma_X$, $\sigma_Z$, $a$ and $b$.

   ![Channel for problem 5](image.png)