Assessment 2

Due: Thursday – 4pm, Gradescope entry code: 948XVG

- Please sign the honor code set up on Gradescope.
- Please upload your answers to Gradescope before 4pm.
- Start a new page for every problem.
- The only allowable aid is a double-sided sheet of notes.
- Questions are weighted differently. The total number of points is 110.

Good luck!
A List of Useful Formulas:

### Product-to-sum

<table>
<thead>
<tr>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \cos \theta \cos \varphi = \cos(\theta - \varphi) + \cos(\theta + \varphi)$</td>
</tr>
<tr>
<td>$2 \sin \theta \sin \varphi = \cos(\theta - \varphi) - \cos(\theta + \varphi)$</td>
</tr>
<tr>
<td>$2 \sin \theta \cos \varphi = \sin(\theta + \varphi) + \sin(\theta - \varphi)$</td>
</tr>
<tr>
<td>$2 \cos \theta \sin \varphi = \sin(\theta + \varphi) - \sin(\theta - \varphi)$</td>
</tr>
</tbody>
</table>

### Sum-to-product

<table>
<thead>
<tr>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tan \theta \tan \varphi = \frac{\cos(\theta - \varphi) - \cos(\theta + \varphi)}{\cos(\theta - \varphi) + \cos(\theta + \varphi)}$</td>
</tr>
</tbody>
</table>

$$
\prod_{k=1}^{n} \cos \theta_k = \frac{1}{2^n} \sum_{e \in S} \cos(e_1 \theta_1 + \cdots + e_n \theta_n)
$$

where $S = \{1, -1\}^n$

Figure 1: Trigonometric identities.

- **Discrete-Time Fourier Transform:**

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>$X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{n-M}$</td>
<td>$e^{-j2\pi f M}$</td>
</tr>
<tr>
<td>$a^n u_n \ (0 &lt;</td>
<td>a</td>
</tr>
<tr>
<td>$W \text{sinc} (W n)$</td>
<td>$\text{rect} \left( \frac{f}{W} \right)$</td>
</tr>
<tr>
<td>$W \text{sinc}^2 (W n)$</td>
<td>$\text{tri} \left( \frac{f}{W} \right)$</td>
</tr>
<tr>
<td>$e^{-j \alpha n}$</td>
<td>$\delta \left( f + \frac{\alpha}{2\pi} \right)$</td>
</tr>
</tbody>
</table>

where $u_n = 0$ for $n < 0$ and 1 otherwise, $\text{tri}(t) = \max \{1 - |t|, 0\}$ and $\text{rect}(t) = \begin{cases}  
1, & |t| < 0.5, \\
0, & \text{else}.
\end{cases}$
• **Kalman Filter Recursion:** Suppose that the state evolves as $X_1 \sim \mathcal{N}(0, \sigma_X^2)$,

$$X_n = \alpha X_{n-1} + W_n, \quad W_n \sim \mathcal{N}(0, \sigma_W^2), \quad n = 2, 3, \ldots$$

and the observation satisfies

$$Y_n = hX_n + Z_n, \quad Z_n \sim \mathcal{N}(0, \sigma_Z^2), \quad n = 1, 2, \ldots$$

where $X_1, W_2 \cdots, Z_1, \cdots$ are independent. Then the Kalman filter recursion:

$$\hat{X}_1 (Y_1) = \frac{h \sigma_X^2 Y_1}{h^2 \sigma_X^2 + \sigma_Z^2}, \quad v_1^2 = \frac{\sigma_X^2 \sigma_Z^2}{h^2 \sigma_X^2 + \sigma_Z^2}. \quad$$

For $n = 2, 3, \ldots$,

$$u_{n-1}^2 = \alpha^2 v_{n-1}^2 + \sigma_W^2;$$

$$\hat{X}_n (Y_1^n) = \alpha \hat{X}_{n-1} (Y_1^{n-1}) + \frac{hu_{n-1}^2}{h^2 u_{n-1}^2 + \sigma_Z^2} \left[ Y_n - h \alpha \hat{X}_{n-1} (Y_1^{n-1}) \right];$$

$$v_n^2 = \frac{u_{n-1}^2 \sigma_Z^2}{h^2 u_{n-1}^2 + \sigma_Z^2},$$

where $Y_1^n = (Y_1, Y_2, \ldots, Y_n)$.

• **Cramer’s rule:**

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \quad$$
1. (10 points)

Consider the detection problem given $n$ sensor observations $Y$:

$$Y = Xh + Wg + Z,$$

where $X$ is equally likely to be 0 or 1, $W \sim N(0, 1)$, $Z \sim N(0, \sigma^2 I)$, and $X, W, Z$ are mutually independent. $\|h\| = \|g\| = 1$. We are interested in detecting $X$ from $Y$.

a) (5 points) For a given $h$, what is an interference direction $g$ that is the worst in terms of the probability of error of detecting $X$?

b) (5 points) For a given $h$, what is an interference direction $g$ that is the best in terms of the probability of error of detecting $X$?

No justification of your answers are needed.
2. (30 points) In detection problems, we consider the MAP and ML detectors. These can be extended to the estimation problem in the natural way. Given $X, Y$ jointly distributed as $f_{X,Y}$:

$$
\hat{X}_{\text{MAP}}(y) = \arg\max_x f_{X|Y}(x|y) \\
\hat{X}_{\text{ML}}(y) = \arg\max_x f_{Y|X}(y|x).
$$

In this question, we compare the performance of these estimators to the MMSE estimator.

a) (15 points) Suppose you observe

$$
Y = X + W,
$$

where $X \sim \mathcal{N}(2, 1)$ and $W \sim \mathcal{N}(0, 1)$ and $X$ and $W$ are independent. You want to estimate $X$. Compute:

i. the MMSE estimate of $X$ given $Y$;

ii. the MAP estimate of $X$ given $Y$;

iii. the ML estimate of $X$ given $Y$.

b) (6 points) Let $e_{\text{mmse}}, e_{\text{map}}, e_{\text{ml}}$ be the mean-square errors of the three estimates in part (a) respectively. Order them. Justify your ordering. (You do not need to provide the actual values of the errors.)

c) (9 points) Suppose you further observe

$$
V = X + Z,
$$

where $Z \sim \mathcal{N}(0, 2)$ and $X, Z, W$ are mutually independent. Order the mean-square errors of the MMSE estimate, MAP estimate and ML estimate of $X$ given $Y$ and $V$. Justify your ordering. (You do not need to provide the actual values of the errors.)
3. (40 points) Consider the scalar dynamical system with noisy observations:

\[
\begin{align*}
X_1 & \sim \mathcal{N}(0, 1) \\
X_n &= \alpha X_{n-1} + \sqrt{1 - \alpha^2} W_n \quad n = 2, 3, \ldots \\
Y_n &= X_n + Z_n \quad n = 1, 2, \ldots
\end{align*}
\]

where \( W_n \sim \mathcal{N}(0, 1) \) and \( Z_n \sim \mathcal{N}(0, 1) \) and \( X_1, W_2, W_3, \ldots, Z_1, Z_2, \ldots \) are mutually independent.

a) (18 points) Compute, for general \( n \geq 2 \):

i. the MMSE estimate of \( X_n \) given \( Y_n \);

ii. the MMSE estimate of \( X_n \) given \( X_{n-1} \);

iii. the MMSE estimate of \( X_n \) given \( Y_{n-1} \);

iv. the MMSE estimate of \( X_n \) given \( Y_1, Y_2, \ldots, Y_{n-1} \);

v. the MMSE estimate of \( X_n \) given \( \hat{X}_{n-1} \), where \( \hat{X}_{n-1} \) is the Kalman filter estimate of \( X_{n-1} \) given \( Y_1, Y_2, \ldots, Y_{n-1} \);

vi. the MMSE estimate of \( X_n \) given \( Y_1, Y_2, \ldots, Y_n \).

b) (14 points) Let \( e_1, e_2, e_3, e_4, e_5, e_6 \) be the mean-squared errors of the 6 estimates of \( X_n \) in part (a) respectively. For each of the pairs \( e_i \) and \( e_j \) below, state whether (A) \( e_i > e_j \); (B) \( e_i < e_j \); (C) \( e_i = e_j \); or (D) not enough information to determine.

i. \( e_1 \) and \( e_2 \);

ii. \( e_2 \) and \( e_3 \);

iii. \( e_1 \) and \( e_3 \);

iv. \( e_1 \) and \( e_4 \);

v. \( e_3 \) and \( e_4 \);

vi. \( e_4 \) and \( e_5 \);

vii. \( e_5 \) and \( e_6 \);

c) (8 points) Your friend claims to have simulated this system and she plotted the resulting sample paths of \( \{X_n\} \) (in blue) and \( \{\hat{X}_n\} \) (in red) for some value of \( \alpha \). (Plots on next page.) She asks you to guess whether the \( \alpha \) value she used is closed to 0 or close to 1. What do you think? Or is she pulling your leg and it is a fake simulation which doesn’t correspond to any \( \alpha \)? Explain.
Figure 2: Your friend’s plots of $X_n$ (blue) and $\hat{X}_n$ (red) over time.
4. (30 points) \( \{X_n\} \) is a zero-mean stationary process with covariance function \( K_X \) and power spectral density \( S_X \). Let \( Y_n = (h \ast X)_n \) and \( Z_n = X_n - Y_n \), as shown in Figure 3, where \( h \) is the impulse response of the system. Let \( H \) be the frequency response of the system.

\[ \{X_n\} \rightarrow h \rightarrow \{Y_n\} - \rightarrow \{Z_n\} \]

Figure 3: System relationships.

a) (4 points) Is \( \{Y_n\} \) stationary? If so, compute the power spectral density of \( \{Y_n\} \). If not, explain why not.

b) (6 points) Is \( \{Z_n\} \) stationary? If so, compute the power spectral density of \( \{Z_n\} \). If not, explain why not.

c) (4 points) Compute \( \text{E}(Z_n^2) \).

d) (8 points) Suppose \( X_n = \cos(2\pi f_0 n + \Theta) \),

where \( f_0 \) is a positive frequency between 0 and 0.5 and \( \Theta \) is uniform between 0 and \( 2\pi \). Compute its power spectral density \( S_X \).

e) (8 points) Suppose \( f_0 \) is unknown. You observe \( \{V_n\} \), where \( V_n = Z_n + W_n \) and \( \{W_n\} \) is zero-mean white noise, independent of \( \{X_n\} \). Explain how you would go about estimating the parameter \( f_0 \). Give a reasonable procedure but there is no need to show that your procedure is optimal in any sense.