PRINT your name: __________________________, __________________________
(last) (first)

All answers should be justified.

You may consult two double-sided sheets of paper with notes. Apart from that, you may not look at books, notes, electronic devices etc.

You have 180 minutes. There are 6 questions, of varying credit (100 points total). The questions are of varying difficulty, so avoid spending too long on any one question.

Good luck!
A List of Useful Formulas:

- **Discrete-Time Fourier Transform:**

<table>
<thead>
<tr>
<th>Time domain $x[n]$</th>
<th>Fourier transform $X(f)$</th>
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</thead>
<tbody>
<tr>
<td>$\delta [n-M]$</td>
<td>$e^{-i2\pi jM}$</td>
</tr>
<tr>
<td>$a^n u[n]$ ($0 &lt;</td>
<td>a</td>
</tr>
<tr>
<td>$W \text{sinc} (Wn)$</td>
<td>$\text{rect} \left( \frac{f}{W} \right)$</td>
</tr>
<tr>
<td>$W \text{sinc}^2 (Wn)$</td>
<td>$\text{tri} \left( \frac{f}{W} \right)$</td>
</tr>
<tr>
<td>$e^{-i\alpha n}$</td>
<td>$\delta (f + \alpha)$</td>
</tr>
</tbody>
</table>

where $\text{tri}(t) = \max \{ 1 - |t|, 0 \}$ and $\text{rect}(t) = \begin{cases} 1, & |t| < 0.5, \\ 0, & \text{else.} \end{cases}$

- **Kalman Filter Recursion:** Suppose that the state evolves as

$$X_{n+1} = \alpha X_n + W_n, \quad W_n \sim \mathcal{N} \left( 0, \sigma_W^2 \right),$$

and the observation satisfies

$$Y_n = hX_n + Z_n, \quad Z_n \sim \mathcal{N} \left( 0, \sigma_Z^2 \right).$$

where $W_1, W_2, \ldots, Z_1, \ldots$ are independent. Then the Kalman filter recursion:

$$\begin{align*}
\hat{x}_n (y_1^{n-1}) &= \alpha \hat{x}_{n-1} (y_1^{n-1}), \\
\sigma^2_{x_{\alpha}} &= \alpha^2 \sigma^2_{x_{\alpha-1}} + \sigma^2_W, \\
\hat{x}_n (y_1^n) &= \hat{x}_n (y_1^{n-1}) + \frac{h \sigma^2_{x_{\alpha}} \left[ y_n - h \hat{x}_n (y_1^{n-1}) \right]}{h^2 \sigma^2_{x_{\alpha}} + \sigma^2_Z}, \\
\sigma^2_{x_{\alpha}} &= \frac{\sigma^2_{x_{\alpha-1}} \sigma^2_Z}{h^2 \sigma^2_{x_{\alpha}} + \sigma^2_Z},
\end{align*}$$

where $\xi_n = X_n - \hat{X}_n (Y_1^n)$, and $\zeta_n = X_n - \hat{X}_n (Y_1^{n-1})$.

- **Cramer’s rule:**

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$
1. Are the following processes stationary? Explain. If so, compute its mean and covariance function. The processes are defined on integer times $n$ from 0 to $\infty$.

a) $\{A_n\}$, with $A_n = 1$ for all $n$.

b) $\{B_n\}$, with $B_n$’s i.i.d. equally probable to be +1 or -1.

c) $\{C_n\}$, such that with probability 0.5, $C_n = +1$ for all $n$, and with probability 0.5, $C_n = -1$ for all $n$.

d) $\{D_n\}$, with $D_n = +1$ for even $n$ and $D_n = -1$ for odd $n$.

e) $\{E_n\}$, with $E_n = -1$ for even $n$ and $E_n = +1$ for odd $n$.

f) $\{F_n\}$, such that with probability 0.5, $F_n = D_n$ for all $n$, and with probability 0.5, $F_n = E_n$ for all $n$.

2. Let $Z_1, Z_2, \cdots$ be a sequence of i.i.d. random variables with distribution Unif[0,1]. Consider the random process defined by $X_n = \sum_{i=1}^{n} Z_i$.

a) Is the process $\{X_n\}$ stationary? Markov?

3. Consider an asymmetrical Mickey Mouse Markov Chain with two states $\{0, 1\}$ and the transition probability $\Pr(X_{n+1} = 1|X_n = 0) = \alpha$ and $\Pr(X_{n+1} = 0|X_n = 1) = \beta$ for some $\alpha, \beta \in (0,1)$.

a) Find the distribution of the initial state $X_0$ such that the process $\{X_n\}$ is stationary.

b) Now let us assume $\alpha = \beta$ and the chain starts at the initial distribution in part (a). Consider the process $\{Y_n\}$ defined by

$$Y_n = X_n + Z_n, \text{ for } n = 0, 1, 2, \cdots$$

where $\{Z_n\}$ are i.i.d. following $\mathcal{N}(0,1)$.

i) What is the distribution of $X_0$?

ii) What is the distribution of $Y_0$?

4. Consider a stochastic process $Z(t) : t \in \mathbb{R}$ for which each sample function is a sequence of rectangular pulses, i.e.

$$Z(t) = \sum_{k=\infty}^{k=-\infty} Z_k \text{rect}(t-k),$$
where

\[ \text{rect}(t) = \begin{cases} 1 & |t| \leq 1/2, \\ 0 & |t| > 1/2, \end{cases} \]

where \( \ldots Z_{-1}, Z_0, Z_1, \ldots \) is a sequence of i.i.d. normal variables \( Z_k \sim \mathcal{N}(0, 1) \).

(a) Is \( \{Z(t) : t \in \mathbb{R}\} \) a Gaussian random process? Explain why or why not.

(b) Find the covariance function \( E[Z(t)Z(\tau)] \) of \( \{Z(t) : t \in \mathbb{R}\} \). (Neglect the case where \( t \) or \( \tau \) are equal to an integer plus 1/2.)

(c) Is \( \{Z(t) : t \in \mathbb{R}\} \) a stationary random process? Explain carefully.

(d) Now suppose the stochastic process is modified by introducing a random time shift \( \Phi \) which is uniformly distributed between 0 and 1. Thus, the new process \( \{V(t) : t \in \mathbb{R}\} \) is defined by \( V(t) = \sum_{k=-\infty}^{k=\infty} Z_k \text{rect}(t - k + \Phi) \). Find the conditional distribution \( V(0.5) \) conditional on \( V(0) = v \).

(e) Is \( \{V(t) : t \in \mathbb{R}\} \) a Gaussian random process? Explain why or why not.

(f) Find the covariance function of \( \{V(t) : t \in \mathbb{R}\} \).

(g) Is \( \{V(t) : t \in \mathbb{R}\} \) a stationary random process? It is easier to explain than to write a lot of equations.

Let \( X(t), Y(t), \) and \( W(t) \) be independent random processes; \( X(t) \) and \( Y(t) \) are zero-mean stationary Gaussian processes with \( R_X(\tau) = R_Y(\tau) = e^{-|\tau|} \). \( W(t) \) is the random telegraph process,

\[ W(t) = A(-1)^{N(t)}, \]

where \( N(t) \) is a Poisson process with parameter \( \lambda \), and the random variable

\[ A = \begin{cases} 1 & \text{with probability 0.5} \\ -1 & \text{with probability 0.5}. \end{cases} \]

\( A \) and \( N(t) \) are independent. Now define the new process \( Z(t) \) as

\[ Z(t) = \begin{cases} X(t) & \text{if } W(t) = 1 \\ Y(t) & \text{if } W(t) = -1. \end{cases} \]

1. Find the first order distribution of \( Z(t) \).

2. Is \( Z(t) \) a Gaussian random process? Justify your answer.

3. Is \( Z(t) \) WSS? Justify your answer.