All answers should be justified.

You may consult two double-sided sheets of paper with notes. Apart from that, you may not look at books, notes, electronic devices etc.

You have 180 minutes. There are 6 questions, of varying credit (100 points total). The questions are of varying difficulty, so avoid spending too long on any one question.

Good luck!
A List of Useful Formulas:

- **Discrete-Time Fourier Transform:**

<table>
<thead>
<tr>
<th>Time domain $x[n]$</th>
<th>Fourier transform $X(f)$</th>
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</thead>
<tbody>
<tr>
<td>$\delta[n-M]$</td>
<td>$e^{-j2\pi M}$</td>
</tr>
<tr>
<td>$a^n u[n]$ ($0 &lt;</td>
<td>a</td>
</tr>
<tr>
<td>$W \text{sinc} ,(Wn)$</td>
<td>$\text{rect} \left( \frac{t}{W} \right)$</td>
</tr>
<tr>
<td>$W \text{sinc}^2 ,(Wn)$</td>
<td>$\text{tri} \left( \frac{t}{W} \right)$</td>
</tr>
<tr>
<td>$e^{-j2\pi t}$</td>
<td>$\delta (f + a)$</td>
</tr>
</tbody>
</table>

where $\text{tri}(t) = \max \{1 - |t|, 0\}$ and $\text{rect}(t) = \begin{cases} 1, & |t| < 0.5, \\ 0, & \text{else}. \end{cases}$

- **Kalman Filter Recursion:** Suppose that the state evolves as

$$X_{n+1} = \alpha X_n + W_n, \quad W_n \sim \mathcal{N} \left( 0, \sigma_w^2 \right),$$

and the observation satisfies

$$Y_n = hX_n + Z_n, \quad Z_n \sim \mathcal{N} \left( 0, \sigma_Z^2 \right).$$

where $W_1, W_2, \ldots, Z_1, \ldots$ are independent. Then the Kalman filter recursion:

$$\begin{align*}
\hat{x}_n \left( y_{n-1} \right) &= \alpha \hat{x}_{n-1} \left( y_{n-1} \right), \\
\sigma_{\hat{x}_n}^2 &= \alpha^2 \sigma_{\hat{x}_{n-1}}^2 + \sigma_w^2, \\
\hat{x}_n \left( y_n \right) &= \hat{x}_n \left( y_{n-1} \right) + \frac{h \sigma_{\hat{x}_n}^2 \left( y_n - h \hat{x}_n \left( y_{n-1} \right) \right)}{h^2 \sigma_{\hat{x}_n}^2 + \sigma_Z^2}, \\
\sigma_{\hat{x}_n}^2 &= \frac{\sigma_{\hat{x}_n}^2 \sigma_Z^2}{h^2 \sigma_{\hat{x}_n}^2 + \sigma_Z^2},
\end{align*}$$

where $\xi_n = X_n - \hat{X}_n \left( Y_1^n \right)$, and $\zeta_n = X_n - \hat{X}_n \left( Y_1^{n-1} \right)$.

- **Cramer’s rule:**

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$
1. Prove the following statements.

1. \(|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}\).
2. \(|R_{XY}(\tau)| \leq \frac{1}{2} (R_X(0) + R_Y(0))\).

2. Let the received signal over an additive noise channel be \(Y(t) = X(t) + Z(t)\). The input signal \(X(t)\) is a WSS process with zero mean and autocorrelation function \(R_X(\tau) = P \cos(10\pi \tau) \cdot \text{sinc}(\tau)\). The noise \(Z(t)\) is a white noise process with power spectral density \(S_Z(f) = N/2, -\infty < f < \infty\). The signal and noise processes are uncorrelated.

Find the transfer function of the best infinite smoothing filter for \(X(t)\) given \(Y(t), -\infty < \tau < \infty\).

Your answers should be in terms only of \(P\) and \(N\).

3. Consider the following variation on the Gauss-Markov process:

\[X_0 \sim \mathcal{N}(0, a)\]
\[X_n = \frac{1}{2} X_{n-1} + Z_n, \quad n \geq 1,\]
where \(Z_1, Z_2, Z_3, \ldots\) are i.i.d. \(\mathcal{N}(0, 1)\) independent of \(X_0\).

1. Find \(a\) such that \(X_n\) is stationary. Find the mean and autocorrelation functions of \(X_n\).

2. Consider the sample mean \(S_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad n \geq 1\). Show that \(S_n\) converges to the process mean in probability even though the sequence \(X_n\) is neither i.i.d. nor uncorrelated.

4. Let \(\{X_n\}\) be a discrete-time continuous-valued Markov random process, that is,

\[f(x_{n+1} | x_1, x_2, \ldots, x_n) = f(x_{n+1} | x_n)\]
for every \(n \geq 1\) and for all sequences \((x_1, x_2, \ldots, x_{n+1})\).

1. Show that \(f(x_1, \ldots, x_n) = f(x_1)f(x_2 | x_1) \cdots f(x_n | x_{n-1}) = f(x_n)f(x_{n-1} | x_n) \cdots f(x_1 | x_2)\).
2. Show that \(f(x_n | x_1, x_2, \ldots, x_k) = f(x_n | x_k)\) for every \(k\) such that \(1 \leq k < n\).
3. Show that \(f(x_{n+1}, x_{n-1} | x_n) = f(x_{n+1} | x_n)f(x_{n-1} | x_n)\), that is, the past and the future are independent given the present.
Consider the random process

\[ X(t) = Z_1 \cos \omega t + Z_2 \sin \omega t, \quad -\infty < t < \infty, \]

where \( Z_1 \) and \( Z_2 \) are i.i.d. discrete random variables such that \( p_{Z_i}(+1) = p_{Z_i}(-1) = \frac{1}{2} \).

1. Is \( X(t) \) wide-sense stationary? Justify your answer.
2. Is \( X(t) \) strict-sense stationary? Justify your answer.