All answers should be justified.

You may consult two double-sided sheets of paper with notes. Apart from that, you may not look at books, notes, electronic devices etc.

You have 180 minutes. There are 6 questions, of varying credit (100 points total). The questions are of varying difficulty, so avoid spending too long on any one question.

Good luck!
A List of Useful Formulas:

• Discrete-Time Fourier Transform:

<table>
<thead>
<tr>
<th>Time domain $x[n]$</th>
<th>Fourier transform $X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta[n-M]$</td>
<td>$e^{-j2\pi fM}$</td>
</tr>
<tr>
<td>$a^n u[n]$ ($0 &lt;</td>
<td>a</td>
</tr>
<tr>
<td>$W \text{sinc}(Wn)$</td>
<td>$\text{rect}\left(\frac{f}{W}\right)$</td>
</tr>
<tr>
<td>$W \text{sinc}^2(Wn)$</td>
<td>$\text{tri}\left(\frac{f}{W}\right)$</td>
</tr>
<tr>
<td>$e^{-j\alpha n}$</td>
<td>$\delta(f + \alpha)$</td>
</tr>
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where $\text{tri}(t) = \max\{1 - |t|, 0\}$ and $\text{rect}(t) = \begin{cases} 1, & |t| < 0.5, \\ 0, & \text{else}. \end{cases}$

• Kalman Filter Recursion: Suppose that the state evolves as

$$X_{n+1} = \alpha X_n + W_n, \quad W_n \sim \mathcal{N}(0, \sigma^2_W),$$

and the observation satisfies

$$Y_n = hX_n + Z_n, \quad Z_n \sim \mathcal{N}(0, \sigma^2_Z).$$

where $W_1, W_2, \ldots, Z_1, \ldots$ are independent. Then the Kalman filter recursion:

$$\begin{align*}
\hat{x}_n(y^n_{1}) &= \alpha \hat{x}_{n-1}(y^{n-1}_{1}), \\
\sigma^2_{x_n} &= \alpha^2 \sigma^2_{x_{n-1}} + \sigma^2_W, \\
\hat{x}_n(y^n_{1}) &= \hat{x}_n(y^{n-1}_{1}) + \frac{h \sigma^2_{x_n} [y_n - h \hat{x}_n(y^{n-1}_{1})]}{h^2 \sigma^2_{x_n} + \sigma^2_Z}, \\
\sigma^2_{x_n} &= \frac{\sigma^2_{x_n} \sigma^2_{Z}}{h^2 \sigma^2_{x_n} + \sigma^2_{Z}},
\end{align*}$$

where $\xi_n = X_n - \hat{X}_n(Y^n_1)$, and $\zeta_n = X_n - \hat{X}_n(Y^{n-1}_1)$.

• Cramer’s rule:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$
1. Prove the following statements.

   1. \(|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}\).
   2. \(|R_{XY}(\tau)| \leq \frac{1}{2}(R_X(0) + R_Y(0))\).

Solution:

1. We use the Schwarz inequality, \(\mathbb{E}(XY)^2 \leq \mathbb{E}(X^2)\mathbb{E}(Y^2)\).

\[
R_{XY}(\tau)^2 = \mathbb{E}(X(t + \tau)Y(t))^2
\leq \mathbb{E}(X(t + \tau)^2)\mathbb{E}(Y(t)^2) \quad \text{(Schwarz inequality)}
= R_X(0)R_Y(0).
\]

Taking the square root of both sides yields \(R_{XY}(\tau) \leq \sqrt{R_X(0)R_Y(0)}\).

2. In the following list of inequalities, each follows from the previous one, except for the first inequality, which is obvious.

\[
\mathbb{E}((X(t + \tau) - Y(t))^2) \geq 0
\]
\[
\mathbb{E}((X(t + \tau))^2) - 2\mathbb{E}(X(t + \tau)Y(t)) + \mathbb{E}(Y(t)^2) \geq 0
\]
\[
\mathbb{E}((X(t + \tau))^2) + \mathbb{E}(Y(t)^2) \geq 2\mathbb{E}(X(t + \tau)Y(t))
\]
\[
R_X(0) + R_Y(0) \geq 2R_{XY}(\tau)
\]
\[
R_{XY}(\tau) \leq \frac{1}{2}(R_X(0) + R_Y(0))
\]

Repeating these steps for \(\mathbb{E}((X(t + \tau) + Y(t))^2)\), we get \(-R_{XY}(\tau) \leq \frac{1}{2}(R_X(0) + R_Y(0))\). Therefore \(|R_{XY}(\tau)| \leq \frac{1}{2}(R_X(0) + R_Y(0))\).
2. Let the received signal over an additive noise channel be $Y(t) = X(t) + Z(t)$. The input signal $X(t)$ is a WSS process with zero mean and autocorrelation function $R_X(\tau) = P \cos(10\pi \tau) \cdot \text{sinc}(\tau)$. The noise $Z(t)$ is a white noise process with power spectral density $S_Z(f) = N/2$, $-\infty < f < \infty$. The signal and noise processes are uncorrelated.

Find the transfer function of the best infinite smoothing filter for $X(t)$ given $Y(\tau)$, $-\infty < \tau < \infty$.

Your answers should be in terms only of $P$ and $N$.

**Solution:** We first find $S_{XY}(f)$ and $S_Y(f)$. Since $X(t)$ and $Z(t)$ are zero mean and uncorrelated

$$R_{XY}(\tau) = R_X(\tau),$$
$$S_{XY}(f) = \frac{P}{2} \left( \text{Rect}(f - 5) + \text{Rect}(f + 5) \right),$$
$$R_Y(\tau) = R_X(\tau) + R_Z(\tau),$$
$$S_Y(f) = \frac{P}{2} \left( \text{Rect}(f - 5) + \text{Rect}(f + 5) \right) + \frac{N}{2}.$$

The transfer function of the best infinite smoothing filter is

$$H(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{P}{N + P} \left( \text{Rect}(f - 5) + \text{Rect}(f + 5) \right).$$
3.

Consider the following variation on the Gauss-Markov process:

\[ X_0 \sim \mathcal{N}(0, a) \]
\[ X_n = \frac{1}{2} X_{n-1} + Z_n, \quad n \geq 1, \]

where \( Z_1, Z_2, Z_3, \ldots \) are i.i.d. \( \mathcal{N}(0, 1) \) independent of \( X_0 \).

1. Find \( a \) such that \( X_n \) is stationary. Find the mean and autocorrelation functions of \( X_n \).

2. Consider the sample mean \( S_n = \frac{1}{n} \sum_{i=1}^{n} X_i \), \( n \geq 1 \). Show that \( S_n \) converges to the process mean in probability even though the sequence \( X_n \) is neither i.i.d. nor uncorrelated.

Solution:

1. We are asked to find \( a \) such that \( \mathbb{E}(X_n) \) is independent of \( n \) and \( R_X(n_1, n_2) \) depends only on \( n_1 - n_2 \).

   For \( X_n \) to be stationary, \( \mathbb{E}(X_n^2) \) must be independent of \( n \). Thus

   \[ \mathbb{E}(X_n^2) = \frac{1}{4} \mathbb{E}(X_{n-1}^2) + \mathbb{E}(Z_n^2) + \mathbb{E}(X_{n-1} Z_n) = \frac{1}{4} \mathbb{E}(X_n^2) + 1. \]

   Therefore, \( a = \mathbb{E}(X_n^2) = \mathbb{E}(X_0^2) = \frac{4}{3} \). Using the method of lecture notes 6, we can easily verify that \( \mathbb{E}(X_n) = 0 \) for every \( n \) and that

   \[ R_X(n_1, n_2) = \mathbb{E}(X_{n_1} X_{n_2}) = \frac{4}{3} 2^{-|n_1 - n_2|}. \]

2. To prove convergence in probability, we first prove convergence in mean square and then use the fact that mean square convergence implies convergence in probability.

   \[ \mathbb{E}(S_n) = \mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(X_i) = \frac{1}{n} \sum_{i=1}^{n} 0 = 0. \]

   To show convergence in mean square we show that \( \text{Var}(S_n) \to 0 \) as \( n \to \infty \).

   \[ \text{Var}(S_n) = \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) = \mathbb{E} \left( \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right)^2 \right) \quad \text{(since } \mathbb{E}(X_i) = 0) \]

   \[ = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} R_X(i, j) = \frac{4}{3n^2} \left( n + 2 \sum_{i=1}^{n-1} (n-i) 2^{-i} \right) \]

   \[ \leq \frac{4}{3n} \left( 1 + 2 \sum_{i=1}^{n-1} 2^{-i} \right) \leq \frac{4}{3n} \left( 1 + 2 \sum_{i=1}^{\infty} 2^{-i} \right) = \frac{4}{n}. \]

   Thus \( S_n \) converges to the process mean, even though the sequence is not i.i.d.
Let \( \{X_n\} \) be a discrete-time continuous-valued Markov random process, that is,
\[
f(x_{n+1} | x_1, x_2, \ldots, x_n) = f(x_{n+1} | x_n)
\]
for every \( n \geq 1 \) and for all sequences \((x_1, x_2, \ldots, x_{n+1})\).

1. Show that \( f(x_1, \ldots, x_n) = f(x_1)f(x_2 | x_1) \cdots f(x_n | x_{n-1}) = f(x_n)f(x_{n-1} | x_n) \cdots f(x_1 | x_2) \).

2. Show that \( f(x_n | x_1, x_2, \ldots, x_k) = f(x_n | x_k) \) for every \( k \) such that \( 1 \leq k < n \).

3. Show that \( f(x_{n+1}, x_{n-1} | x_n) = f(x_{n+1} | x_n)f(x_{n-1} | x_n) \), that is, the past and the future are independent given the present.

**Solution:**

1. We are given that \( f(x_{n+1} | x_1, x_2, \ldots, x_n) = f(x_{n+1} | x_n) \). From the chain rule, in general,
\[
f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2 | x_1)f(x_3 | x_1, x_2) \cdots f(x_n | x_1, x_2, \ldots, x_{n-1}).
\]
Thus, by the definition of Markovity,
\[
f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2 | x_1)f(x_3 | x_2) \cdots f(x_n | x_{n-1}).
\]
(1)

Similarly, applying the chain rule in reverse we get
\[
f(x_1, x_2, \ldots, x_n) = f(x_n)f(x_{n-1} | x_n)f(x_{n-2} | x_{n-1}, x_n) \cdots f(x_1 | x_2, x_3, \ldots, x_n).
\]
Next,
\[
f(x_i | x_{i+1}, \ldots, x_n) = \frac{f(x_i, x_{i+1}, \ldots, x_n)}{f(x_{i+1}, \ldots, x_n)} = \frac{f(x_i)f(x_{i+1} | x_i)}{f(x_{i+1})} = f(x_i | x_{i+1}),
\]
(2)
where the second equality follows from (1). Therefore
\[
f(x_1, x_2, \ldots, x_n) = f(x_n)f(x_{n-1} | x_n)f(x_{n-2} | x_{n-1}, x_n) \cdots f(x_1 | x_2, x_3, \ldots, x_n)
\]
\[
= f(x_n)f(x_{n-1} | x_n)f(x_{n-2} | x_{n-1}) \cdots f(x_1 | x_2),
\]
where the second line follows from (2).

2. First consider
\[
f(x_n | x_1, \ldots, x_k) = \frac{f(x_1, \ldots, x_k, x_n)}{f(x_1, \ldots, x_k)}
\]
\[
= \frac{f(x_n)f(x_k | x_n)f(x_{k-1} | x_k, x_n) \cdots f(x_1 | x_2, \ldots, x_k, x_n)}{f(x_k)f(x_{k-1} | x_k) \cdots f(x_1 | x_2)},
\]
(3)
where the denominator in the second line follows from part (a). Next consider
\[
f(x_{k-1}, \ldots, x_n) = f(x_k, x_n)f(x_{k-1} | x_k, x_n)f(x_{k+1}, x_{k+2}, \ldots, x_n | x_{k-1}, x_k, x_n)
\]
\[
= f(x_n)f(x_{n-1} | x_n) \cdots f(x_{k-1} | x_k),
\]

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where the second line follows from (2). Integrating both sides over \(x_{k+1}, \ldots, x_{n-1}\) (i.e., using the law of total probability), we get

\[ f(x_k, x_n) f(x_{k-1} | x_k, x_n) = f(x_k, x_n) f(x_{k-1} | x_k). \]

Finally, substituting into (3), we get

\[
f(x_n | x_1, \ldots, x_k) = \frac{f(x_n) f(x_k | x_n) f(x_{k-1} | x_k) \cdots f(x_1 | x_2)}{f(x_k) f(x_{k-1} | x_k) \cdots f(x_1 | x_2)}
= \frac{f(x_n) f(x_k | x_n)}{f(x_k)} = f(x_n | x_k).
\]

3. By the chain rule for conditional densities,

\[
f(x_{n+1}, x_{n-1} | x_n) = f(x_{n+1} | x_n) f(x_{n-1} | x_{n+1}, x_n) = f(x_{n+1} | x_n) f(x_{n-1} | x_n),
\]

where the second equality follows from (2).
Consider the random process

\[ X(t) = Z_1 \cos \omega t + Z_2 \sin \omega t, \quad -\infty < t < \infty, \]

where \( Z_1 \) and \( Z_2 \) are i.i.d. discrete random variables such that \( p_{Z_i}(1) = p_{Z_i}(-1) = \frac{1}{2} \).

1. Is \( X(t) \) wide-sense stationary? Justify your answer.

2. Is \( X(t) \) strict-sense stationary? Justify your answer.

Solution:

1. We first check the mean.

\[ \mathbb{E}(X(t)) = \mathbb{E}(Z_1 \cos \omega t + \mathbb{E}(Z_2 \sin \omega t) = 0 \cdot \cos(\omega t) + 0 \cdot \sin(\omega t) = 0. \]

The mean is independent of \( t \). Next we consider the autocorrelation function.

\[
\mathbb{E}(X(t + \tau)X(t)) = \mathbb{E}((Z_1 \cos(\omega(t + \tau)) + Z_2 \sin(\omega(t + \tau))) (Z_1 \cos(\omega t) + Z_2 \sin(\omega t)))
\]

\[
= \mathbb{E}(Z_1^2) \cos(\omega(t + \tau)) \cos(\omega t) + \mathbb{E}(Z_2^2) \sin(\omega(t + \tau)) \sin(\omega t)
\]

\[
= \cos(\omega(t + \tau)) \cos(\omega t) + \sin(\omega(t + \tau)) \sin(\omega t)
\]

\[
= \cos(\omega(t + \tau) - \omega t) = \cos \omega \tau.
\]

The autocorrelation function is also time invariant. Therefore \( X(t) \) is WSS.

2. Note that \( X(0) = Z_1 \cos 0 + Z_2 \sin 0 = Z_1 \), so \( X(0) \) has the same pmf as \( Z_1 \). On the other hand,

\[ X(\pi/(4\omega)) = Z_1 \cos(\pi/4) + Z_2(\sin \pi/4) \]

\[ = \frac{1}{\sqrt{2}}(Z_1 + Z_2) \]

\[ \sim \begin{cases} 
\frac{1}{4} & x = \pm \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\frac{1}{2} & x = 0 \\
0 & \text{otherwise}
\end{cases} \]

This shows that \( X(\pi/(4\omega)) \) does not have the same pdf or even same range as \( X(0) \). Therefore \( X(t) \) is not first-order stationary and as a result is not SSS.