1. **Shot noise channel.** Consider an additive noise channel with input signal $X \sim U(0, 1)$ and output signal $Y = X + Z$, where the noise $Z|X = x \sim \mathcal{N}(0, ax)$, for some constant $a > 0$, i.e., the noise variance is proportional to the signal. Observing $Y$, find the MMSE linear estimate of $X$. Your answer should be in terms of only $a$ and $Y$.

2. **Additive-noise channel with path gain.** Consider the output $Y$ of an additive-noise channel with path gain, where $X$ and $Z$ are zero mean and uncorrelated, and $a$ and $b$ are constants. Find the MMSE linear estimate of $X$ given $Y$ and its MSE in terms only of $\sigma_X, \sigma_Z, a$ and $b$.

![Figure 1: Channel for problem 2](image)

3. **Camera measurement.** The measurement from a camera can be expressed as $Y = AX + Z$, where $X$ is the object position with mean $\mu$ and variance $\sigma_X^2$, $A$ is the occlusion indicator function and is equal to 1 (if the camera can see the object) with probability $p$, and 0 (if the camera cannot see the object) with probability $(1 - p)$, and $Z$ is the measurement error with mean 0 and variance $\sigma_Z^2$. Assume that $X$, $A$, and $Z$ are independent. Find the best linear MSE estimate of $X$ given the camera measurement $Y$. Your answer should be in terms of only $\mu$, $\sigma_X^2$, $\sigma_Z^2$, and $p$.

4. **Jointly Gaussian random variables.** Let $X$ and $Y$ be jointly Gaussian random variables with mean 0 and covariance matrix

$$
\begin{bmatrix}
\sigma_X^2 & \sigma_X \sigma_Y \rho_{X,Y} \\
\sigma_X \sigma_Y \rho_{X,Y} & \sigma_Y^2
\end{bmatrix}
$$

a. What is the pdf of $E(X | Y)$?

b. What is the minimum MSE estimate of $Y^2$ given $X$?

Your answers should be in terms of $\sigma_X, \sigma_Y, \rho_{X,Y}$, and the random variables $X$ and $Y$.

5. **Estimation vs. detection.** Signal $X$ and noise $Z$ are independent random variables, where

$$
X = \begin{cases} 
+1 & \text{with probability } \frac{1}{2} \\
-1 & \text{with probability } \frac{1}{2},
\end{cases}
$$

and $Z \sim U[-2, +2]$. Their sum $Y = X + Z$ is observed.

a. Find the minimum MSE estimate of $X$ given $Y$ and the corresponding mean square error. What is the probability of error of this estimate?
b. Suppose that we decide whether $X = +1$ or $X = -1$ using a decoder that minimizes the probability of error. Find this optimal decoder and its probability of error. Compare the optimal decoder’s MSE to the minimum MSE.

6. Jointly Gaussian random variables, redux. Consider the following joint pdf for $X$ and $Y$:

$$f_{X,Y}(x, y) = \frac{1}{\pi^{3/4}} e^{-\frac{1}{2}(\frac{1}{4}x^2 + \frac{1}{4}y^2 + \frac{3}{4}xy - 8x - 16y + 16)}$$

a. Find $E(X)$, $E(Y)$, $\text{Var}(X)$, $\text{Var}(Y)$, and $\text{Cov}(X, Y)$.

b. Find the minimum MSE estimate of $X$ given $Y$ and the corresponding MSE.

7. Mean square error estimation. The number of packets arriving per unit time at a node in a communication network is a Poisson random variable $X$ with rate $\Lambda \sim \text{Exp}(a)$. Find the MMSE estimate of the rate $\Lambda$ given the observation $X$. Your answer should be in terms only of $X$ and the constant $a$. Hint: you do not need to evaluate complicated integrals here. Just use integration by parts, i.e., $\int_\alpha^\beta udv = uv|_\beta^\alpha - \int_\alpha^\beta v du$. The final result will look very nice!

The following problems are optional and need not be turned in for grading.

1. Independence vs. Conditional Independence Give an example of random variables $X, Y,$ and $Z$ where $f_{X,Z}(x, z) = f_X(x)f_Z(z)$ but $f_{X,Z|Y}(x, z|y) \neq f_{X|Y}(x|y)f_{Z|Y}(z|y)$ i.e. independence does not imply conditional independence.

2. Sum and difference. Let $X$ and $Y$ be two random variables, and define $U = X - Y$ and $V = X + Y$. Find the minimum MSE linear estimate of $V$ given $U$ as a function of the random variables and $E(X)$, $E(Y)$, $\sigma_X$, $\sigma_Y$, $\rho_{X,Y}$, where $\sigma_X = \sqrt{\text{Var}(X)}$, $\rho_{X,Y} = \text{corr}(X,Y)$.

3. Covariance matrices. Which of the following matrices can be a covariance matrix? Justify your answer. Either construct a random vector $\mathbf{X}$ with the given covariance matrix as a function of the i.i.d. zero mean unit variance random variables $Z_1, Z_2, Z_3$, or establish a contradiction as was done in lecture.

\[
\begin{pmatrix}
1 & 2 \\
0 & 2
\end{pmatrix} \quad \begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix} \quad \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3
\end{pmatrix} \quad \begin{pmatrix}
1 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 3
\end{pmatrix}
\]

4. Conditional Independence does not imply Independence. In class, we saw an example in which two independent, identically distributed random variables conditioned on a third random variable were no longer independent. Here, we examine an example of the opposite case: is it possible for conditionally independent random variables to be not independent?

Suppose $X_3 \sim \text{U}[0,1]$, and given $X_3$: $X_1, X_2 \text{i.i.d. } \sim \text{Bern}(X_3)$. Show that $X_1, X_2$ are not independent, although they are conditionally independent given $X_3$ as given. Work out the joint distribution $P_{X_1, X_2}$.

Hint: the Beta function is $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$. 
