1. **Poisson Processes.** Consider two independent Poisson processes $N_1(t), N_2(t)$ with the same rate $\lambda$. Define the process $N(t) = N_1(t) - N_2(t)$, and we’ll refer to $N_1(t)$ as the “arrivals” process, and $N_2(t)$ as the “departures” process.
   a. Draw, on the same plot, a typical sample path of $N_1(t)$ and $N_2(t)$.
   b. For the above sample paths, draw the corresponding sample paths of $N(t)$ and, for comparison, the sample path of process $N_1(t) + N_2(t)$.
   c. Is $N(t)$ an independent increment process? Justify your answer.
   d. Find the distribution of $T_1$, the time of the first change (arrival or departure) of the process $N(t)$. (Hint: the probability $P(T_1 \geq t)$ can be interpreted as the probability that at time $t$ there have not yet been any arrivals nor departures.)
   e. Find the mean and autocorrelation functions of $N(t)$.
   f. Find the MMSE linear estimator of $N(t)$ based on the sample $N(t_1)$ when:
      i. $t < t_1$
      ii. $t > t_1$

2. **Stationary Gauss-Markov process.** Consider the following variation on the Gauss-Markov process:
   
   $X_0 \sim \mathcal{N}(0, a)$
   
   $X_n = \frac{1}{2}X_{n-1} + Z_n$, \quad $n \geq 1$, 

   where $Z_1, Z_2, Z_3, \ldots$ are i.i.d. $\mathcal{N}(0, 1)$, independent of $X_0$.
   a. Find the mean and autocorrelation functions of $X_n$.
   b. Find $a$ such that $X_n$ is wide sense stationary.

3. **Sawtooth process.** Let $X(t) = g(t - T)$, where $g(t)$ is the periodic triangular waveform shown in Figure 1 and the delay $T$ is a random variable with $T \sim U[0, 1]$. Is $X(t)$ a strict-sense stationary random process? Justify your answer.

4. **Windowed Poisson process.** Let $N(t)$, for $t \geq 0$, be a Poisson process with rate $\lambda > 0$, and let $X(t)$, for $t \geq 0$, be defined by $X(t) = N(t + 1) - N(t)$. Thus, $X(t)$ is the number of events of $N$ during the time window $(t, t + 1]$.
   a. Sketch a typical sample path of $N$, and the corresponding sample path of $X$.
   b. Find the mean function $\mu_X(t)$, for $t \geq 0$ and the autocorrelation function $R_X(t_1, t_2)$ for $t_1, t_2 \geq 0$. Express your answer in a simple form.
c. Is $X$ a Markov process? Why or why not?

5. Modified telegraph process. Let $X(t), Y(t),$ and $W(t)$ be independent random processes; $X(t)$ and $Y(t)$ are zero-mean stationary Gaussian processes with $R_X(\tau) = R_Y(\tau) = e^{-|\tau|}$. $W(t)$ is the random telegraph process,

$$W(t) = A(-1)^{N(t)},$$

where $N(t)$ is a Poisson process with parameter $\lambda$, and the random variable

$$A = \begin{cases} 
1 & \text{with probability 0.5} \\
-1 & \text{with probability 0.5}.
\end{cases}$$

$A$ and $N(t)$ are independent. Now define the new process $Z(t)$ as

$$Z(t) = \begin{cases} 
X(t) & \text{if } W(t) = 1 \\
Y(t) & \text{if } W(t) = -1.
\end{cases}$$

a. Find the first order distribution of $Z(t)$.

b. Is $Z(t)$ a Gaussian random process? Justify your answer.

c. Is $Z(t)$ WSS? Justify your answer.

6. Generating a random process with a prescribed PSD. The power spectral density $S_X(f)$ of every WSS process is real, even, and nonnegative. In this problem you will show that, conversely, if $S(f)$ is a real, even, nonnegative function such that $\int_{-\infty}^{\infty} S(f)df < \infty$, then $S(f)$ is the PSD for some WSS random process. Let us consider the case that

$$\int_{-\infty}^{\infty} S(f)df = 1.$$ 

Define the random process

$$X(t) = \cos(2\pi Ft + \Theta),$$
where $F \sim S(f)$ and $\Theta \sim U[0, 2\pi)$ are independent.

a. Show that $X(t)$ is WSS.

b. Find the power spectral density of $X(t)$. Interpret the result.

c. Consider the power spectral density

$$S(f) = \frac{\alpha}{\alpha^2 + (\pi f)^2}, \quad -\infty < f < \infty.$$  

Use MATLAB (or any other programming language) to generate sample functions of $X(t)$ for $\alpha = 1, 5, 20$. 