1. **Switched RC circuit.** Consider the circuit in Figure 1. The voltage source $V(t)$ models the thermal noise in the resistor. At time $t = 0$, the switch is closed. Compute the average power $E[V_o^2(t)]$ as a function of time $t$.

![Switched RC circuit](image)

Figure 1: Switched RC circuit

Lecture Notes 8 slides 15 and 16 show the computation of the average output noise power of an RC circuit. Compare that result to your answer in this problem.

2. **Prediction.** Consider the wide sense stationary process $Y(n)$ defined by

$$Y(n) = aY(n - 1) + X(n)$$

where $X(n)$ is a zero-mean white noise process with variance $\sigma^2$, and $|a| < 1$. We are interested in predicting the future of the process $l$ time steps into the future, i.e. to predict $Y(n + l)$ based on $Y(m), -\infty < m \leq n$.

a. Prove that the optimal linear predictor is

$$\hat{Y}(n + l|n) = a^l Y(n),$$

and explain why this predictor does not need the measurements that proceed $Y(n)$. Hint: use the orthogonality principle.

b. Express, in terms of $a, \sigma^2$ and $l$, the prediction error

$$\epsilon^2_l = E[(Y(n + l) - \hat{Y}(n + l|n))^2].$$

Check and explain your results for $l = 1$ and $l \to \infty$.

c. Define the $l$th order innovation process by

$$U_l(n) = Y(n + l) - \hat{Y}(n + l|n).$$

Prove that

$$E[U_l(n)U_l(n - k)] = 0, \quad k \geq l,$$

i.e. the autocorrelation sequence of $U_l(n)$ satisfies $R_{U_l}(k) = 0$, for all $k \geq l$. 

3. **LTI system with WSS process input.** Let $Y(t) = h(t) * X(t)$ and $Z(t) = X(t) - Y(t)$, as shown in Figure 2.

a. Find $S_Z(f)$.

b. Find $E[Z^2(t)]$.

Your answers should be in terms of $S_X(f)$ and the transfer function $H(f) = \mathcal{F}\{h(t)\}$.

![Figure 2: Linear time-invariant system](image)

4. **Linear system with feedback.** Given the system in Figure 3 where $X(t)$ and $Z(t)$ are independent zero-mean WSS random processes, and $h(t)$ is the impulse response of a linear time invariant system, find the power spectral density of $Y(t)$.

![Figure 3: Linear system with feedback](image)

5. **Discrete-time LTI system with white noise input.** Let $\{X_n : -\infty < n < \infty\}$ be a discrete-time white noise process, i.e., $E[X_n] = 0$ and

$$R_X(n) = \begin{cases} 
1 & n = 0 \\
0 & \text{otherwise.}
\end{cases}$$

The process is filtered using a linear time-invariant system with impulse response

$$h(n) = \begin{cases} 
\alpha & n = 0 \\
\beta & n = 1 \\
0 & \text{otherwise.}
\end{cases}$$

Find $\alpha$ and $\beta$ such that the output process $Y_n$ has

$$R_Y(n) = \begin{cases} 
2 & n = 0 \\
1 & |n| = 1 \\
0 & \text{otherwise.}
\end{cases}$$

6. **Narrow-band process over additive white noise channel.** Let the received signal over an additive noise channel be $Y(t) = X(t) + Z(t)$. The input signal $X(t)$ is a WSS process with zero
mean and autocorrelation function $R_X(\tau) = P \cos(10\pi \tau) \cdot \text{sinc}(\tau)$. The noise $Z(t)$ is a white noise process with power spectral density $S_Z(f) = N/2, \ -\infty < f < \infty$. The signal and noise processes are uncorrelated.

a. Find and sketch the transfer function of the best infinite smoothing filter for $X(t)$ given $Y(\tau), \ -\infty < \tau < \infty$.

b. Find the MSE of the best infinite smoothing filter.

Your answers should be in terms of only $P$ and $N$.

7. Linear predictor with even coefficients. Consider the problem of prediction for a process $Y(n)$ with the power spectral density

$$S_Y(f) = \begin{cases} 
1 & \text{if } |f| < 1/4 \\
1/2 & \text{if } |f| \geq 1/4 
\end{cases}$$

in the interval $[-1/2, 1/2]$. Suppose the linear predictor is restricted to even coefficients only, i.e. $a_1 = a_3 = a_5 = \cdots = 0$, hence

$$\hat{Y}(n|n-1) = a_2 Y(n-2) + a_4 Y(n-4) + \cdots$$

Find the optimal coefficients $a_2, a_4, \ldots$. What is the variance of the associated prediction error?

**Hint:** Compute the autocorrelation function of $Y(n)$, denoted by $R_Y(k)$, and look at even values of $k$. 