Homework #9 Solutions

Please submit your assignment as a PDF to the class Gradescope page.

1. **Switched RC circuit.** Consider the circuit in Figure 1. The voltage source $V(t)$ models the thermal noise in the resistor. At time $t = 0$, the switch is closed. Compute the average power $E[V_o^2(t)]$ as a function of time $t$.

![Switched RC circuit](image)

Figure 1: Switched RC circuit

Lecture Notes 8 slides 15 and 16 show the computation of the average output noise power of an RC circuit. Compare that result to your answer in this problem.

**Solution**

Let us lump the voltage source and the switch together as a (non-stationary) source that drives the linear time-invariant RC system. The autocorrelation function of this source is

$$R_V(t_1, t_2) = \begin{cases} 2kTR\delta(t_1 - t_2) & \text{if } t_1, t_2 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The impulse response of the LTI system composed of the resistor and the capacitor is

$$h(t) = \frac{1}{RC}e^{-t/(RC)}u(t).$$

The output random process is given by the convolution

$$V_o(t) = \int_{-\infty}^{\infty} h(\tau)V(t - \tau)d\tau.$$
We can compute the output power as a function of $t$ as

$$E[V_o^2(t)] = E \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) V(t - \tau_1) V(t - \tau_2) d\tau_2 d\tau_1 \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_V(t - \tau_1, t - \tau_2) d\tau_2 d\tau_1$$

$$= 2kTR \int_{-\infty}^{t} h(\tau_1) h(\tau_1) d\tau_1$$

$$= \frac{2kT}{(RC)^2} \int_{0}^{t} e^{-2\tau_1/(RC)} d\tau_1$$

$$= \frac{2kT}{RC} \left[ -\frac{RC}{2} e^{-2\tau_1/(RC)} \right]_0^t$$

$$= \frac{kT}{C} (1 - e^{-2t/(RC)}).$$

Note that as $t \to \infty$, the average output power converges to $kT/C$.

2. Prediction. Consider the wide sense stationary process $Y(n)$ defined by

$$Y(n) = aY(n-1) + X(n)$$

where $X(n)$ is a zero-mean white noise process with variance $\sigma^2$, and $|a| < 1$. We are interested in predicting the future of the process $l$ time steps into the future, i.e. to predict $Y(n+l)$ based on $Y(m), -\infty < m \leq n$.

a. Prove that the optimal linear predictor is

$$\hat{Y}(n+l|n) = a^l Y(n),$$

and explain why this predictor does not need the measurements that proceed $Y(n)$. Hint: use the orthogonality principle.

b. Express, in terms of $a, \sigma^2$ and $l$, the prediction error

$$\epsilon_l^2 = E[(Y(n+l) - \hat{Y}(n+l|n))^2].$$

Check and explain your results for $l = 1$ and $l \to \infty$.

c. Define the $l$th order innovation process by

$$U_l(n) = Y(n+l) - \hat{Y}(n+l|n).$$

Prove that

$$E[U_l(n)U_l(n-k)] = 0, \quad k \geq l,$$

i.e. the autocorrelation sequence of $U_l(n)$ satisfies $R_{U_l}(k) = 0$, for all $k \geq l$.

Solution

a. The linear optimal predictor of $\hat{Y}(n+l)$ based on all $Y(m), -\infty < m \leq n$ is of course a weighted linear combination of the $Y(m)$, i.e.

$$\hat{Y}(n+l|n) = \sum_{k \geq 0} \alpha_k Y(n-k), \quad m \geq 0.$$
Using the orthogonality principle, we can write for all $m \geq 0$
\[
E \left[ \left( Y(n + l) - \sum_{k \geq 0} \alpha_k Y(n - k) \right) Y(n - m) \right] = 0,
\]
which results in the equality
\[
R_Y(m + l) = \sum_{k \geq 0} \alpha_k R_Y(k - m), \quad m \geq 0.
\]

To find the optimal $\alpha_k$, we first find the autocorrelation function. Note that because $Y(n)$ is recursive, we find the autocorrelation function as follows, if $m \geq 0$:
\[
R_Y(m) = E[Y(n + m)Y(n)] \\
= E[(aY(n + m - 1) + X(n + m))Y(n)] \\
= E[(a(aY(n + m - 2) + X(n + m - 1)) + X(n + m))Y(n)] \\
= E[(a^2Y(n + m - 2) + aX(n + m - 1) + X(n + m))Y(n)] \\
\vdots \\
= E \left[ (a^m Y(n) + \sum_{k=0}^{m-1} a^k X(n + m - k))Y(n) \right] \\
= a^m R_Y(0).
\]

For any $m$, $R_Y(m) = a^{\lfloor m \rfloor} R_Y(0)$. With this expression, it is easy to see that the equality above is satisfied for all $m \geq 0$ if $\alpha_0 = a^l$, and $\alpha_k = 0$ for all $k > 0$. Thus, $\hat{Y}(n + l|n) = \sum_{k \geq 0} \alpha_k Y(n - k) = a^l Y(n)$.

b. The prediction error is
\[
\epsilon^2 = E[(Y(n + l) - \hat{Y}(n + l|n))^2] \\
= E[(Y(n + l))^2] - 2a^l E[Y(n + l)Y(n)] + a^{2l} E[Y(n)^2] \\
= (1 + a^{2l}) R_Y(0) - 2a^l R_Y(l)
\]

To find $R_Y(0)$ we write
\[
R_Y(0) = E[Y(n)Y(n)] \\
= E[(aY(n - 1) + X(n))(aY(n - 1) + X(n))] \\
= a^2 R_Y(0) + \sigma^2 \\
\Rightarrow R_Y(0) = \frac{\sigma^2}{1 - a^2}.
\]

Plugging this back into the prediction error, we find that
\[
\epsilon^2 = \sigma^2 \frac{1 - a^{2l}}{1 - a^2}.
\]

Note that the orthogonality principle could have been invoked after the first equality, resulting in the same prediction error.
For $l = 1$, the prediction error variance is equal to the prediction error variance of a first-order autoregressive process based on its past, and for $l = \infty$, it is equal to the variance of the process.

c. We note that $U_l(n)$ is the error process due to predicting $Y(n_l)$, and thus it is orthogonal to all the observations used in its prediction. Since $U_l(n - k)$ for $k \geq l$ is a linear combination of these observations, $U_l(n)$ and $U_l(n - k)$ are orthogonal for $k \geq l$.

3. \textit{LTI system with WSS process input.} Let $Y(t) = h(t) * X(t)$ and $Z(t) = X(t) - Y(t)$, as shown in Figure 2.

a. Find $S_Z(f)$.
b. Find $E[Z^2(t)]$.

Your answers should be in terms of $S_X(f)$ and the transfer function $H(f) = \mathcal{F}\{h(t)\}$.

![Figure 2: Linear time-invariant system](image)

**Solution**

a. To find $S_z(f)$, we start with

$$Z(t) = (\delta(t) - h(t)) * X(t).$$

Taking the power spectral density on both sides, we arrive at

$$S_Z(f) = |1 - H(f)|^2 S_X(f).$$

b. The average power of $Z(t)$ is the area under $S_Z(f)$,

$$E[Z^2(t)] = \int_{-\infty}^{\infty} |1 - H(f)|^2 S_X(f) df.$$ 

4. \textit{Linear system with feedback.} Given the system in Figure 3 where $X(t)$ and $Z(t)$ are independent zero-mean WSS random processes, and $h(t)$ is the impulse response of a linear time invariant system, find the power spectral density of $Y(t)$.

![Figure 3: Linear system with feedback](image)
Solution

It follows from the block diagram that
\[(X(t) - Y(t)) * h(t) + Z(t) = Y(t).\]

Sorting terms, this is equivalent to
\[X(t) * h(t) + Z(t) = (\delta(t) + h(t)) * Y(t).\]

Taking the power spectral density on both sides, this implies
\[S_X(f)|H(f)|^2 + S_Z(f) = |1 + H(f)|^2 S_Y(f),\]
where we have used the fact that \(X(t)\) and \(Z(t)\) are independent, and the power spectral density of the sum is thus the sum of the power spectral densities. Finally, we conclude that
\[S_Y(f) = \frac{|H(f)|^2}{|1 + H(f)|^2} S_X(f) + \frac{1}{|1 + H(f)|^2} S_Z(f).\]

5. **Discrete-time LTI system with white noise input.** Let \(\{X_n : -\infty < n < \infty\}\) be a discrete-time white noise process, i.e., \(E[X_n] = 0\) and
\[R_X(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}.\]

The process is filtered using a linear time-invariant system with impulse response
\[h(n) = \begin{cases} \alpha & n = 0 \\ \beta & n = 1 \\ 0 & \text{otherwise} \end{cases}.\]

Find \(\alpha\) and \(\beta\) such that the output process \(Y_n\) has
\[R_Y(n) = \begin{cases} 2 & n = 0 \\ 1 & |n| = 1 \\ 0 & \text{otherwise} \end{cases}.\]

**Solution**

We are given that \(R_X(n)\) is a discrete-time unit impulse. Therefore
\[R_Y(n) = h(n) * R_X(n) * h(-n) = h(n) * h(-n).\]

The impulse response \(h(n)\) is the sequence \((a, \beta, 0, 0, \ldots)\). The convolution with \(h(-n)\) has only finitely many nonzero terms.
\[R_Y(0) = 2 = h(0) * h(0) = \alpha^2 + \beta^2\]
\[R_Y(+1) = 1 = h(1) * h(-1) = \alpha \beta\]
\[R_Y(-1) = 1 = R_Y(1)\]

This pair of equations has two solutions: \(\alpha = +1\) and \(\beta = +1\) or \(\alpha = -1\) and \(\beta = -1\).

6. **Narrow-band process over additive white noise channel.** Let the received signal over an additive noise channel be \(Y(t) = X(t) + Z(t)\). The input signal \(X(t)\) is a WSS process with zero
mean and autocorrelation function \( R_X(\tau) = P \cos(10\pi \tau) \cdot \text{sinc}(\tau) \). The noise \( Z(t) \) is a white noise process with power spectral density \( S_Z(f) = N/2, -\infty < f < \infty \). The signal and noise processes are uncorrelated.

a. Find and sketch the transfer function of the best infinite smoothing filter for \( X(t) \) given \( Y(\tau), -\infty < \tau < \infty \).

b. Find the MSE of the best infinite smoothing filter.

Your answers should be in terms of only \( P \) and \( N \).

**Solution**

a. We first find \( S_{XY}(f) \) and \( S_Y(f) \). Since \( X(t) \) and \( Z(t) \) are zero mean and uncorrelated,

\[
R_{XY}(\tau) = R_X(\tau),
\]

\[
S_{XY}(f) = \frac{P}{2}(\text{Rect}(f - 5) + \text{Rect}(f + 5)),
\]

\[
R_Y(\tau) = R_X(\tau) + R_Z(\tau),
\]

\[
S_Y(f) = \frac{P}{2}(\text{Rect}(f - 5) + \text{Rect}(f + 5)) + \frac{N}{2}.
\]

The transfer function of the best infinite smoothing filter is

\[
H(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{P}{N + P}(\text{Rect}(f - 5) + \text{Rect}(f + 5)).
\]

This is sketched in Figure 4.

![Figure 4: Transfer function \( H(f) \)](image)

b. The MSE is given by

\[
\text{MSE} = \int_{-\infty}^{\infty} \left( S_X(f) - \left( \frac{S_{XY}(f)}{S_Y(f)} \right)^2 \right) df
\]

\[
= 2 \int_{-5.5}^{5.5} \left( \frac{P}{2} - \frac{(P/2)^2}{N/2 + P/2} \right) df
\]

\[
= \frac{NP}{P + N}
\]
7. **Linear predictor with even coefficients.** Consider the problem of prediction for a process $Y(n)$ with the power spectral density

$$S_Y(f) = \begin{cases} 
1 & \text{if } |f| < 1/4 \\
1/2 & \text{if } |f| \geq 1/4 
\end{cases}$$

in the interval $[-1/2, 1/2]$. Suppose the linear predictor is restricted to even coefficients only, i.e. $a_1 = a_3 = a_5 = \cdots = 0$, hence

$$\hat{Y}(n|n-1) = a_2 Y(n-2) + a_4 Y(n-4) + \cdots$$

Find the optimal coefficients $a_2, a_4, \ldots$. What is the variance of the associated prediction error?

**Hint:** Compute the autocorrelation function of $Y(n)$, denoted by $R_Y(k)$, and look at even values of $k$.

**Solution**

We note that

$$R_Y(k) = \frac{1}{2} \delta(k) + \frac{\sin(k\pi/2)}{k\pi}$$

which is 0 for all non-zero even values of $k$. Thus, $Y(n)$ is uncorrelated with all of the observations when the set of observations is restricted to samples of even distance from $Y(n)$. Hence, the prediction error variance is equal to the process variance, which is the area of the power spectral density given by $3/4$. 