1. **Poisson Processes.** Consider two independent Poisson processes $N_1(t), N_2(t)$ with the same rate $\lambda$. Define the process $N(t) = N_1(t) - N_2(t)$, and we’ll refer to $N_1(t)$ as the “arrivals” process, and $N_2(t)$ as the “departures” process.

   a. Draw, on the same plot, a typical sample path of $N_1(t)$ and $N_2(t)$.

   b. For the above sample paths, draw the corresponding sample paths of $N(t)$ and, for comparison, the sample path of process $N_1(t) + N_2(t)$.

   c. Is $N(t)$ an independent increment process? Justify your answer.

   d. Find the distribution of $T_1$, the time of the first change (arrival or departure) of the process $N(t)$. (Hint: the probability $P(T_1 \geq t)$ can be interpreted as the probability that at time $t$ there have not yet been any arrivals nor departures.)

   e. Find the mean and autocorrelation functions of $N(t)$.

   f. Find the MMSE linear estimator of $N(t)$ based on the sample $N(t_1)$ when:
      
      i. $t < t_1$
      
      ii. $t > t_1$

**Solution**

a. Plot should contain two non-identical Poisson process trajectories.

b. $N(t)$ should take discrete, real values (can be negative and/or zero) corresponding to changes in $N_1(t)$ relative to $N_2(t)$, and vice versa. $N_1(t) + N_2(t)$ should be monotonically increasing, and takes on discrete positive values corresponding to changes in $N_1(t)$ relative to $N_2(t)$, and vice versa.

c. Yes, $N(t)$ is an independent increment process. This can be seen using the following arguments:
   
   • If $X(t)$ is an independent increment random process, then $-X(t)$ is also an independent increment process.
   
   • The sum of two independent increment processes $X_1(t), X_2(t)$ is also an independent increment process.
   
   • Finally,

   $$P(X_1(t_b) - X_2(t_b) - X_1(t_1) - X_2(t_a)|X_1(t_a), X_2(t_a))$$

   $$\leftrightarrow P(X_1(t_b) - X_2(t_b) - X_1(t_1) - X_2(t_a)|X_1(t_a) - X_2(t_a))$$

   since $X_1(t_a) - X_2(t_a)$ is a function of $X_1(t_1), X_2(t_1)$, and if a random variable is independent of another random variable, then the random variable is also independent of a function of the other random variable.
Alternatively, the independent increment process may be seen through writing
the distribution of \( N(t) \):

\[
P(N(t_2) - N(t_1) = n) = P((N_1(t_2) - N_1(t_1)) - (N_2(t_2) - N_2(t_1)) = n)
\]

\[
= \sum_m P((N_1(t_2) - N_1(t_1)) = m + n)(N_2(t_2) - N_2(t_1)) = m) \times
P((N_2(t_2) - N_2(t_1)) = m)
\]

\[
= \sum_m P((N_1(t_2) - N_1(t_1)) = m + n)P((N_2(t_2) - N_2(t_1)) = m)
\]

\[
= \sum_m e^{-\lambda(t_2-t_1)}[\lambda(t_2 - t_1)]^{n+m} e^{-\lambda(t_2-t_1)}[\lambda(t_2 - t_1)]^m
(n-m)! (m)!
\]

which depends only on the current time interval \( t_2 - t_1 \), so \( N(t) \) is an independent increment process.

d. Using the hint, we write:

\[
P(T_1 \geq t) = P(N_1(t) = 0, N_2(t) = 0) = P(N_1(t) = 0)P(N_2(t) = 0)
\]

since \( N_1(t), N_2(t) \) independent

\[
= e^{-2\lambda t} \text{ since } N_1(t), N_2(t) \text{ have same rate } \lambda,
\]

which we use to find the CDF of \( T_1 \) as

\[
F_{T_1}(t) = P(T_1 \leq t)
= 1 - P(T_1 \geq t)
= 1 - e^{-2\lambda t}.
\]

Now we can find the distribution of \( T_1 \) as

\[
f_{T_1}(t) = \frac{d}{dt} F_{T_1}(t)
= 2\lambda e^{-2\lambda t},
\]

i.e. \( T_1 \) is distributed as an exponential random variable with parameter \( 2\lambda \).

e. The mean function of \( N(t) \) is

\[
\mu_N(t) = E[N(t)]
= E[N_1(t) - N_2(t)]
= E[N_1(t)] - E[N_2(t)]
= 0
\]

since \( N_1(t), N_2(t) \) are i.i.d. Poisson random processes.
The autocorrelation function of $N(t)$ is
\[
R_N(t_1, t_2) = E[(N_1(t_1) - N_2(t_1))(N_1(t_2) - N_2(t_2))]
\]
\[
= E[N_1(t_1)N_1(t_2)] + E[N_2(t_1)N_2(t_2)] - E[N_2(t_1)]E[N_1(t_2)] - E[N_1(t_1)]E[N_2(t_2)]
\]
\[
= 2(\lambda \min\{t_1, t_2\} + \lambda^2 t_1 t_2) - 2\lambda^2 t_1 t_2
\]
\[
= 2\lambda \min\{t_1, t_2\},
\]
where the third equality follows from the autocorrelation function of a Poisson process with rate $\lambda$ derived in lecture.

f. The LMMSE estimator of $N(t)$ based on sample $N(t_1)$ is of course
\[
\hat{N}(t) = \frac{\text{Cov}(N(t),N(t_1))}{\text{Var}(N(t_1))}N(t_1)
\]
\[
= \frac{R_N(t, t_1)}{R_N(t_1, t_1)}N(t_1)
\]
\[
= \frac{\min\{t, t_1\}}{t_1}N(t_1)
\]

since $\mu_N(t) = 0$. So, the LMMSE estimators for the different cases are
i. $t < t_1$: $\hat{N}(t) = \frac{t}{t_1}N(t_1)$
ii. $t > t_1$: $\hat{N}(t) = N(t_1)$

2. **Stationary Gauss-Markov process.** Consider the following variation on the Gauss-Markov process:

\[
X_0 \sim N(0, a)
\]
\[
X_n = \frac{1}{2}X_{n-1} + Z_n, \quad n \geq 1,
\]
where $Z_1, Z_2, Z_3, \ldots$ are i.i.d. $N(0, 1)$, independent of $X_0$.

a. Find the mean and autocorrelation functions of $X_n$.
b. Find $a$ such that $X_n$ is wide sense stationary.

**Solution**

a. Using the method of lecture notes 6, we can easily verify that $E(X_n) = 0$ for every $n$ and that
\[
R_X(n_1, n_2) = E(X_{n_1}X_{n_2}) = 2^{-|n_1-n_2|}\left[\frac{4}{3} + \left(\frac{1}{4}\right)^{\max(n_1,n_2)} \left(a - \frac{4}{3}\right)\right].
\]
b. We are asked to find $a$ such that $R_X(n_1, n_2)$ depends only on $n_1 - n_2$. Thus $a = \frac{4}{3}$.

Alternatively, for $X_n$ to be wide sense stationary, $E(X^2_n)$ must be independent of $n$. Thus
\[
E(X^2_n) = \frac{1}{3} E(X^2_{n-1}) + E(Z^2_n) + E(X_{n-1}Z_n) = \frac{1}{3} E(X^2_n) + 1.
\]
Therefore, $a = E(X_0^2) = E(X_n^2) = \frac{4}{3}$.

3. **Sawtooth process.** Let $X(t) = g(t-T)$, where $g(t)$ is the periodic triangular waveform shown in Figure 1 and the delay $T$ is a random variable with $T \sim U[0, 1]$.

Is $X(t)$ a strict-sense stationary random process? Justify your answer.

**Solution**

By definition, $X(t)$ is stationary if the joint distribution of any set of samples does not depend on the placement of the time origin. This means that the joint cdf of $X(t_1), \ldots, X(t_k)$ is the same as that of $X(t_1 + \tau), \ldots, X(t_k + \tau)$, for all $t_1, t_2, \ldots, t_k$ and all $\tau$. First we show that the first-order cdf of $X(t)$ is independent of time. If $0 \leq x \leq 1$, then

$$F_{X(t)}(x) = P\{X(t) \leq x\} = P\{g(t - \Delta) \leq x\}$$

$$= \int_0^1 P\{g(t - \Delta) \leq x \mid \Delta = \tau\} f_\Delta(\tau) d\tau$$

$$= \int_0^1 P\{g(t - \Delta) \leq x \mid \Delta = \tau\} d\tau$$

$$= \int_0^1 P\{g(t - \tau) \leq x\} d\tau \quad \text{since } g(t) \text{ has period 1}$$

$$= \int_0^1 P\{g(\tau) \leq x\} d\tau = \int_0^1 P\{(1 - \tau) \leq x\} d\tau$$

$$= \int_0^1 P\{\tau \geq 1 - x\} d\tau = \int_{1-x}^1 1 d\tau = 1 - (1 - x) = x.$$  

In other words, $X(t)$ is uniformly distributed between 0 and 1 for every $t$.

To show that higher order distributions are time invariant, note that $g(t)$ can be expressed in terms of a given $g(t_1)$ as $g(t) = (g(t_1) - (t - T)) \mod 1$, where $a \mod b = a - b\lfloor a/b \rfloor$. Thus, given $X(t_1) = x_1$, the sample function of the process is determined
unambiguously, and its values at $t_2, t_3, \ldots, t_k$ depend only on the relative positions of the $t_i$'s. Thus $X(t)$ is strict-sense stationary.

4. *Windowed Poisson process.* Let $N(t)$, for $t \geq 0$, be a Poisson process with rate $\lambda > 0$, and let $X(t)$, for $t \geq 0$, be defined by $X(t) = N(t + 1) - N(t)$. Thus, $X(t)$ is the number of events of $N$ during the time window $(t, t + 1]$.

a. Sketch a typical sample path of $N$, and the corresponding sample path of $X$.

b. Find the mean function $\mu_X(t)$, for $t \geq 0$ and the autocorrelation function $R_X(t_1, t_2)$ for $t_1, t_2 \geq 0$. Express your answer in a simple form.

c. Is $X$ a Markov process? Why or why not?

**Solution**

a. An example with $\lambda = 1$ is shown in the figure below.

![Graph of N(t) and X(t)](image-url)

![Graph of X(t)](image-url)
b. We can compute the mean function as follows

\[ \mu_X(t) = E[X(t)] = E[N(t + 1)] - E[N(t)] = \lambda(t + 1) - \lambda t = \lambda. \]

The autocorrelation function is

\[ R_X(t_1, t_2) = E[X(t_1)X(t_2)] = E[(N(t_1 + 1) - N(t_1))(N(t_2 + 1) - N(t_2))] \]

\[ = R_N(t_1 + 1, t_2 + 1) - R_N(t_1 + 1, t_2) - R_N(t_1, t_2 + 1) + R_N(t_1, t_2) \]

\[ = \lambda(\min\{t_1 + 1, t_2 + 1\} - \min t_1 + 1, t_2 - \min\{t_1, t_2 + 1\} + \min\{t_1, t_2\}) \]

\[ + \lambda^2((t_1 + 1)(t_2 + 1) - (t_1 + 1)t_2 - t_1(t_2 + 1) + t_1t_2) \]

\[ = \begin{cases} \lambda(1 - |t_1 - t_2|) + \lambda^2 & \text{if } |t_1 - t_2| \leq 1, \\ \lambda^2 & \text{otherwise.} \end{cases} \]

where the last step can be seen by considering the four cases

\[ t_1 \in [0, t_2 - 1), \]

\[ t_1 \in [t_2 - 1, t_2), \]

\[ t_1 \in [t_2, t_2 + 1), \quad \text{and} \]

\[ t_1 \in [t_2 + 1, \infty) \]

one by one and computing the min terms for each of them. \textbf{Remark:} Since \( \mu_X(t) \) does not depend on \( t \) and \( R_X(t_1, t_2) \) depends only on \( |t_1 - t_2| \), it follows that \( X(t) \) is WSS.

c. \( X(t) \) is not a Markov process. To prove this, we need to find times \( t_1 < t_2 < t_3 \) and values \( x_1, x_2, x_3 \) such that

\[ P\{X(t_3) = x_3 \mid X(t_2) = x_2, X(t_1) = x_1\} \neq P\{X(t_3) = x_3 \mid X(t_2) = x_2\}. \quad (1) \]

There are several choices that work. We will use \( t_1 = 0, t_2 = 0.5, t_3 = 1.0, \) and \( x_1 = 0, x_2 = 1, x_3 = 0. \) Define the random variables

\[ A = N(0.5) - N(0), \]

\[ B = N(1.0) - N(0.5), \]

\[ A = N(1.5) - N(1.0), \]

\[ A = N(2.0) - N(1.5), \]

each of which is Poisson-distributed with mean \( \lambda/2 \). The four random variables are independent from each other since the time intervals do not overlap. The
right hand side of (1) is then
\[
P\{C + D = 0 \mid B + C = 1\} = \frac{P\{C = 0, D = 0, B + C = 1\}}{P\{B + C = 1\}}
= \frac{P\{C = 0, D = 0, B = 1\}}{P\{B + C = 1\}}
= \frac{P\{C = 0\}P\{D = 0\}P\{B = 1\}}{P\{B + C = 1\}}
= \frac{\lambda/2e^{-(3/2)\lambda}}{\lambda e^{-\lambda}}
= \frac{1}{2}e^{-\lambda/2}.
\]

The left hand side of (1) is
\[
P\{C + D = 0 \mid B + C = 1, A + B = 0\} = P\{C = 0, D = 0 \mid A = 0, B = 0, C = 1\}
= 0
\]
which is different from the right hand side for every \(\lambda\). Therefore, the process \(X(t)\) is not a Markov process for any \(\lambda\).

5. Modified telegraph process. Let \(X(t), Y(t),\) and \(W(t)\) be independent random processes; \(X(t)\) and \(Y(t)\) are zero-mean stationary Gaussian processes with \(R_X(\tau) = R_Y(\tau) = e^{-|\tau|}\). \(W(t)\) is the random telegraph process,
\[
W(t) = A(-1)^{N(t)},
\]
where \(N(t)\) is a Poisson process with parameter \(\lambda\), and the random variable
\[
A = \begin{cases} 
1 & \text{with probability } 0.5 \\
-1 & \text{with probability } 0.5.
\end{cases}
\]

\(A\) and \(N(t)\) are independent. Now define the new process \(Z(t)\) as
\[
Z(t) = \begin{cases} 
X(t) & \text{if } W(t) = 1 \\
Y(t) & \text{if } W(t) = -1.
\end{cases}
\]
a. Find the first order distribution of \(Z(t)\).
b. Is \(Z(t)\) a Gaussian random process? Justify your answer.
c. Is \(Z(t)\) WSS? Justify your answer.

Solution

a. We know that
\[
Z(t) = \begin{cases} 
X(t) & \text{if } W(t) = +1 \\
Y(t) & \text{if } W(t) = -1.
\end{cases}
\]
We can write
\[
P\{Z(t) \leq z\} = P\{W(t) = +1\}P\{Z(t) \leq z | W(t) = +1\} \\
+ P\{W(t) = -1\}P\{Z(t) \leq z | W(t) = -1\} \\
= \frac{1}{2}(P\{X(t) \leq z | W(t) = +1\} + P\{Y(t) \leq z | W(t) = -1\}) \\
= \frac{1}{2}(\phi(z) + \phi(z)) \\
= \phi(z),
\]
where we have used the fact that \(X(t)\) and \(Y(t)\) are independent of \(W(t)\), and where \(\phi\) is the cdf of the normal distribution. Thus, \(Z(t) \sim \mathcal{N}(0, 1)\), independent of \(t\).

b. We consider the joint distribution of two samples \(Z(t_1)\) and \(Z(t_2)\), where \(t_1 < t_2\). Their joint cdf is
\[
P\{Z(t_1) \leq z_1, Z(t_2) \leq z_2\} \\
= \sum_{w_1, w_2 \in \{+1, -1\}} P\{W(t_1) = w_1, W(t_2) = w_2\} \\
\cdot P\{Z(t_1) \leq z_1, Z(t_2) \leq z_2 | W(t_1) = w_1, W(t_2) = w_2\}. \tag{1}
\]
If \(Z(t_1)\) and \(Z(t_2)\) are from the same process (either both from \(X(t)\) or both from \(Y(t)\)), then their joint distribution is
\[
\begin{bmatrix} Z(t_1) \\ Z(t_2) \end{bmatrix} \mid \{W(t_1) = W(t_2)\} \sim \mathcal{N} \left( \begin{bmatrix} 0, & 1 \\ -e^{-(t_2-t_1)}, & 1 \end{bmatrix} \right), \tag{2}
\]
where we have used the autocorrelation function of \(X(t)\) and \(Y(t)\). If they are from different processes (one from \(X(t)\) and one from \(Y(t)\)), then their joint distribution is
\[
\begin{bmatrix} Z(t_1) \\ Z(t_2) \end{bmatrix} \mid \{W(t_1) = W(t_2)\} \sim \mathcal{N} \left( \begin{bmatrix} 0, & 0 \\ 0, & 1 \end{bmatrix} \right), \tag{3}
\]
since \(X(t)\) and \(Y(t)\) are independent. Let \(\Phi_1\) be the cdf of (2), and \(\Phi_2\) the cdf of (3). Substituting into (1), the joint cdf of \(Z(t_1)\) and \(Z(t_2)\) is
\[
P\{Z(t_1) \leq z_1, Z(t_2) \leq z_2\} \\
= P\{W(t_1) = W(t_2)\} \Phi_1(z_1, z_2) + P\{W(t_1) \neq W(t_2)\} \Phi_2(z_1, z_2) \\
= \frac{1 + e^{-2\lambda(t_2-t_1)}}{2} \Phi_1(z_1, z_2) + \frac{1 - e^{-2\lambda(t_2-t_1)}}{2} \Phi_2(z_1, z_2)
\]
since \(W(t_1) = W(t_2)\) only when the number of events \(N(t_2) - N(t_1)\) in interval \((t_1, t_2]\) is even, and \(W(t_1) \neq W(t_2)\) only when the number of events \(N(t_2) - N(t_1)\) in interval \((t_1, t_2]\) is odd, and the probability for each those cases is a Taylor series that converges, respectively, to hyperbolic cosine and sine functions. Taking a linear combination of two different Gaussian cdfs does not result in a Gaussian
cdf. (This is different from taking a linear combination of two Gaussian random variables.) Hence, the joint distribution of \( Z(t_1) \) and \( Z(t_2) \) is not Gaussian, and \( Z(t) \) is not a GRP.

c. It follows from part (a) that the mean and variance of \( Z(t) \) are independent of \( t \). To decide whether \( Z(t) \) is WSS, we only need to compute the correlation function \( R_Z(t_1, t_2) \). In part (b), we already computed the joint distribution of \( Z(t_1) \) and \( Z(t_2) \) for \( t_1 < t_2 \). The joint pdf is

\[
f_{Z(t_1), Z(t_2)}(z_1, z_2) = c_1 \phi_1(z_1, z_2) + c_2 \phi_2(z_1, z_2),
\]

where

\[
c_1 = \frac{1 + e^{-2\lambda (t_2 - t_1)}}{2},
\]

\[
c_2 = \frac{1 - e^{-2\lambda (t_2 - t_1)}}{2},
\]

\[
\phi_1(z_1, z_2) = \mathcal{N}\left(0, \begin{bmatrix} 1 & e^{-(t_2-t_1)} \\ e^{-(t_2-t_1)} & 1 \end{bmatrix}\right),
\]

\[
\phi_2(z_1, z_2) = \mathcal{N}\left(0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).
\]

The correlation function is

\[
R_Z(t_1, t_2) = E[Z(t_1)Z(t_2)] = \int \int z_1 z_2 f_{Z(t_1), Z(t_2)}(z_1, z_2) \, dz_1 \, dz_2
\]

\[
= c_1 \int \int z_1 z_2 \phi_1(z_1, z_2) \, dz_1 \, dz_2 + c_2 \int \int z_1 z_2 \phi_2(z_1, z_2) \, dz_1 \, dz_2
\]

\[
= c_1 e^{-(t_2-t_1)} + c_2 \cdot 0
\]

\[
= \frac{1 + e^{-2\lambda (t_2-t_1)}}{2} e^{-(t_2-t_1)}
\]

This is a function only of \( t_2 - t_1 \), not of \( t_1 \) and \( t_2 \) individually. Thus \( Z(t) \) is WSS.

6. **Generating a random process with a prescribed PSD.** The power spectral density \( S_X(f) \) of every WSS process is real, even, and nonnegative. In this problem you will show that, conversely, if \( S(f) \) is a real, even, nonnegative function such that \( \int_{-\infty}^{\infty} S(f) \, df < \infty \), then \( S(f) \) is the PSD for some WSS random process. Let us consider the case that

\[
\int_{-\infty}^{\infty} S(f) \, df = 1.
\]

Define the random process

\[
X(t) = \cos(2\pi F t + \Theta),
\]

where \( F \sim S(f) \) and \( \Theta \sim U[0, 2\pi) \) are independent.
a. Show that $X(t)$ is WSS.

b. Find the power spectral density of $X(t)$. Interpret the result.

c. Consider the power spectral density

$$S(f) = \frac{\alpha}{\alpha^2 + (\pi f)^2}, \quad -\infty < f < \infty.$$ 

Use MATLAB (or any other programming language) to generate sample functions of $X(t)$ for $\alpha = 1, 5, 20$.

**Solution**

a. First we find the mean of $X(t)$:

$$E(X(t)) = E[\cos(2\pi F t + \Theta)]$$

$$= E_F \left[ E_{\Theta \mid F}(\cos(2\pi F t)) \mid F \right]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} S(f) \int_{-\infty}^{\infty} \cos(2\pi f t + \theta) d\theta d\tau$$

$$= 0$$

since the integral of a periodic function over a full period is 0.

We now find the autocorrelation function:

$$R_X(t_1, t_2) = E_F \left[ E_{\Theta \mid F}(\cos(2\pi F t_1 + \Theta) \cos(2\pi F t_2 + \Theta)) \mid F \right]$$

$$= E_F \left[ \frac{1}{2} \cos(2\pi F(t_1 - t_2)) \right]$$

$$= E_F \left[ \frac{1}{2} \cos(2\pi F \tau) \right]$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} S(\alpha) \cos(2\pi \alpha \tau) \, d\alpha.$$ 

Since the mean is time invariant and the autocorrelation function depends only on the time difference, $X(t)$ is WSS.

b. The power spectral density of $X$ is

$$S_X(f) = \frac{1}{2} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} S(\alpha) \cos(2\pi \alpha \tau) \, d\alpha \right) e^{-j2\pi f \tau} \, d\tau$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} S(\alpha) \int_{-\infty}^{\infty} \cos(2\pi \alpha \tau) e^{-j2\pi f \tau} \, d\tau \, d\alpha$$

$$= \frac{1}{4} (S(f) + S(-f)) \quad \text{by the Fourier transform of cosine}$$

$$= \frac{1}{2} S(f) \quad \text{(since $S(f)$ is even)}$$

Notice that the psd of $X(t)$ is a scalar multiple of the pdf of $F$. This constructively
proves that any function that is real, even, and nonnegative and has a finite integral can be a power spectral density for a WSS random process.

c. The pdf and cdf of the frequency random variable $F$ are related as follows:

$$f_F(f) = S(f) = \frac{\alpha}{\alpha^2 + (\pi f)^2} = \frac{d}{df} F_F(f).$$

Therefore

$$F_F(f) = \frac{1}{2} + \frac{1}{\pi} \arctan \left( \frac{\pi f}{\alpha} \right).$$

Thus the frequency random variable $F$ is a function of a uniform $Z \sim U[0, 1]$.

$$F = F_F^{-1}(Z) = \frac{\alpha}{\pi} \tan(\pi(Z - \frac{1}{2})).$$

% The following MATLAB code generates 5 sample functions of X(t) % for each of three values of alpha: 1, 5 and 20.

```matlab
alpha_list = [ 1 5 20 ];
t = 0:.001:1;
for alpha_index = 1 : 3
    alpha = alpha_list( alpha_index );
    Z = rand( 5, 1 );
    F = (alpha/pi)*tan( pi*( Z - 1/2 ) );
    Theta = pi*( 2*rand( 5, 1 ) - 1 );
    X = cos( 2*pi*F*t + repmat( Theta, 1, 1001 ) );
    subplot( 3, 1, alpha_index );
    plot( t, X );
end

Sample output is shown in the figure below.