Homework #7

Due **Tuesday**, November 28 at **4 p.m.**

Please drop the assignment in the Homework In box in the EE 278 drawer of the class file cabinet on the 2nd floor of the Packard Building.

1. *Absolute value random walk.* Let X_n be a random walk defined by $X_0 = 0, X_n = \sum_{i=1}^{n} Z_i, n \ge 1$, where $\{Z_i\}$ is an i.i.d. process with $\mathrm{P}\{Z_1 = -1\} = \mathrm{P}\{Z_1 = +1\} = \frac{1}{2}$. Define the absolute value random process $Y_n = |X_n|$.

   a. Find $\mathrm{P}\{Y_n = k\}$.

   b. Find $\mathrm{P}\{\max\{Y_i : 1 \le i < 20\} = 10 \,|\, Y_{20} = 0\}$.

2. *Random walk with random start.* Let X_0 be a random variable with pmf

$$p_{X_0}(x) = \begin{cases} \frac{1}{5} & x \in \{-2, -1, 0, +1, +2\} \\ 0 & \text{otherwise} \end{cases}$$

   Suppose that X_0 is the starting position of a random walk $\{X_n : n \ge 0\}$ defined by

$$X_n = X_0 + \sum_{i=1}^{n} Z_i,$$

   where $\{Z_i\}$ is an i.i.d. random process with $\mathrm{P}(Z_1 = -1) = \mathrm{P}(Z_1 = +1) = \frac{1}{2}$ and every Z_i is independent of X_0.

   a. Does X_n have independent increments? Justify your answer.

   b. What is the conditional pmf of X_0 given that $X_{11} = 2$?

3. *Digital modulation using PSK.* The data to be modulated, $\{X_n : n \ge 0\}$, is modeled by a Bernoulli process with p $= 1/2$. Define the discrete-time phase process $\{\Theta_n : n \ge 0\}$ by

$$\Theta_n = \begin{cases} +\frac{\pi}{2} & \text{if } X_n = 1 \\ -\frac{\pi}{2} & \text{if } X_n = 0 \end{cases}$$

   Let $T > 0$ be the transmission time for one bit. Define the continuous-time phase process $\{\Theta(t) : t \ge 0\}$ by $\Theta(t) = \Theta_n$, $nT \le t < (n+1)T$.
   The modulated data is given by

$$X(t) = \cos\left(\frac{4\pi t}{T} + \Theta(t)\right), \quad t \ge 0.$$

   (For the Matlab parts below, download `pskModulation.m` from the course website.)

   a. Use Matlab to sketch a sample function of the process $X(t)$. Let $T = 1$.

b. Find the first order pmf of $X(t)$.

c. Define the process

$$Y(t) = \cos\left(\frac{4\pi t}{T} + \Theta(t) + \Psi\right),$$

where $\Psi \sim \mathrm{U}[0, 2\pi]$ is independent of X_n for $n \geq 0$ and $t \geq 0$. Use MATLAB to sketch a sample path of $Y(t)$. Again let $T = 1$.

d. Find the first-order pdf of $Y(t)$.

4. *Packet arrival.* (This problem comes from the example on slide 6-21). Packets arrive at a router according to a Poisson process $N(t)$ with rate λ. Assume the service time for each packet $T \sim \mathrm{Exp}(\beta)$ is independent of $N(t)$ and of each other. What is the probability that k packets arrive during a service time?

5. *Poisson process branching.* Let $N(t)$ be a Poisson process with rate λ. We split $N(t)$ into two counting subprocesses $N_1(t)$ and $N_2(t)$ such that $N(t) = N_1(t) + N_2(t)$ as follows: each event is randomly and independently assigned to process $N_1(t)$ with probability p, otherwise it is assigned to $N_2(t)$. Prove that $N_1(t)$ is a Poisson process with rate $p\lambda$ and $N_2(t)$ is a Poisson process with rate $(1-p)\lambda$.

6. *Interarrival time process.* Let $N(t)$ be a Poisson process with rate λ. The interarrival time process is defined as $X_n = T_n - T_{n-1}$ for $n \geq 1$, where T_n is the arrival time of the n^{th} event of $N(t)$. Prove that $X_n \sim \mathrm{Exp}(\lambda)$.

7. *Autocorrelation functions.* Find the autocorrelation functions of

a. random binary waveform $X(t)$:
In a digital communication channel, the symbol "1" is represented by the fixed duration rectangular pulse

$$g(t) = \begin{cases} 1 & \text{for } 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

and the symbol "0" is represented by $-g(t)$. The data transmitted over the channel is represented by the random process

$$X(t) = \sum_{k=0}^{\infty} A_k g(t - kT), \quad t \geq 0,$$

where the data $A_0, A_1, A_2, \ldots$ is an i.i.d. random sequence with

$$A_i = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

b. moving average process Y_n:
Let $\{X_n \colon n \geq 1\}$ be a discrete-time white Gaussian noise process, that is, $X_1, X_2, X_3 \ldots$

are i.i.d. random variables with $X_n \sim \mathcal{N}(0, N)$. Consider the *moving-average* process $\{Y_n : n \geq 2\}$ defined by

$$Y_n = \tfrac{2}{3} X_{n-1} + \tfrac{1}{3} X_{n-2}\,, \quad n \geq 2\,.$$

with X_0 defined to be 0.

8. *Poisson process probabilities.* Consider a Poisson process with intensity $\lambda > 0$.

   a. Find the probability that there is (exactly) one arrival in each of the three intervals $[0, 1]$, $[1, 2]$, and $[2, 3]$.

   b. Find the probability that there are two arrivals in the interval $[0, 2]$ and two arrivals in the interval $[1, 3]$. (Note: your answer should be larger than the answer to part (a)).

   c. Find the probability that there are two arrivals in the interval $[1, 2]$ given that there are two arrivals in $[0, 2]$ and two arrivals in $[1, 3]$.