Homework 1

Due: Friday, Sept-25-2020, 5pm – Gradescope entry code: 948XVG

Please upload your answers timely to Gradescope. Start a new page for every problem. For the programming/simulation questions you can use any reasonable programming language. Comment your source code and include the code and a brief overall explanation with your answers.

1. Exercise 1.27 in text.

Exercise 1.27  A computer system has $n$ users, each with a unique name and password. Due to a software error, the $n$ passwords are randomly permuted internally (i.e., each of the $n!$ possible permutations is equally likely). Only those users lucky enough to have had their passwords unchanged in the permutation are able to continue using the system.

(a) What is the probability that a particular user, say user 1, is able to continue using the system?

(b) What is the expected number of users able to continue using the system? Hint: Let $X_i$ be a rv with the value 1 if user $i$ can use the system and 0 otherwise.

Solution:

(a) The obvious (and perfectly correct) solution is that user 1 is equally likely to be assigned any one of the $n$ passwords in the permutation, and thus has probability $1/n$ of being able to continue using the system.

This might be somewhat unsatisfying since it does not reason directly from the property that all $n!$ permutations are equally likely. Thus a more direct argument is that the number of permutations starting with 1 is the number of permutations of the remaining $(n-1)$ users, i.e., $(n-1)!$. These are equally likely, each with probability $1/n!$, so the probability that user 1 (or any given user $i$) can continue to use the system is $(n-1)!/n! = 1/n$.

(b) For the $X_i$ given in the hint, we have already seen that $\Pr(X_i = 1) = 1/n$ and $\Pr(X_i = 0) = 1 - 1/n$. Thus $E[X_i] = 1/n$. The number of users who can continue to use the system is $S_n = \sum_{i=1}^{n} X_i$. From the linearity of expectation, $E[S_n] = \sum_{i=1}^{n} E[X_i] = n/n = 1$.

It is important to understand here that the $X_i$ are not independent, and that the expected value of a finite sum of rv’s is equal to the sum of the expected values whether or not the rv’s are independent.
2. Exercise 1.38 in text (Correction to the question: $0 < \alpha < 1$ instead of just $\alpha < 1$).

**Exercise 1.38** Let $\{X_n; n \geq 1\}$ be a sequence of independent but not identically distributed rv's. We say that the WLLN holds for this sequence if for all $\epsilon > 0$

$$\lim_{n \to \infty} \Pr \left\{ \left| \frac{S_n}{n} - \frac{E[S_n]}{n} \right| \geq \epsilon \right\} = 0,$$

where $S_n = X_1 + X_2 + \cdots + X_n$. (WLLN).

(a) Show that the WLLN holds if there is some constant $A$ such that $\sigma_{X_n}^2 \leq A$ for all $n$.

(b) Suppose that $\sigma_{X_n}^2 \leq An^{1-\alpha}$ for some $\alpha < 1$ and for all $n$. Show that the WLLN holds in this case.

**Solution:**

(a) The variance of $S_n/n$ is $\sigma_{S_n/n}^2 = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_{X_i}^2 \leq nA/n^2 = A/n$. Using the Chebyshev inequality on $S_n/n$,

$$\Pr \left\{ \left| \frac{S_n}{n} - \frac{E[S_n]}{n} \right| \geq \epsilon \right\} \leq \frac{A}{n\epsilon^2}$$

The quantity on the right approaches 0 as $n \to \infty$, for all $\epsilon > 0$, thus the WLLN holds in this case.

(b) This is similar to (a), but it is considerably stronger, since it says that the variance of the $X_i$ can be slowly increasing in $i$, and if this increase is bounded as above, the WLLN still holds. $An^{1-\alpha}$ in increasing in $n$ (since $\alpha < 1$). Hence $\sigma_{X_i}^2 \leq Ai^{1-\alpha} \leq An^{1-\alpha}$ for $i \leq n$.

$$\sigma_{S_n/n}^2 = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_{X_i}^2 \leq \frac{1}{n^2} nAn^{1-\alpha} \leq An^{-\alpha}$$

Using the Chebyshev inequality on $S_n/n$,

$$\Pr \left\{ \left| \frac{S_n}{n} - \frac{E[S_n]}{n} \right| \geq \epsilon \right\} \leq \frac{An^{-\alpha}}{\epsilon^2}$$

which approaches 0 as $n \to \infty$ for all $\epsilon > 0$ (since $\alpha > 0$). Thus the WLLN holds.

3. The Law of Large Numbers is said to hold for a sequence of random variables $S_1, S_2, S_3, S_4, \ldots$ if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \Pr(\left| \frac{1}{n}S_n - E[\frac{1}{n}S_n] \right| > \epsilon) = 0.$$ 

In class we have shown that the Law of Large Numbers holds if $S_n = X_1 + \cdots + X_n$, where the $X_i$'s are i.i.d. random variables. This problem explores if the Law of Large Numbers holds under other circumstances.
Packets are sent from a source to a destination node over the Internet. Each packet is sent on a certain route. Each route has a failure probability of $p$ and different routes fail independently. If a route fails, all packets sent along that route are lost. You can assume that the routing protocol has no knowledge of which route fails.

For each of the following routing protocols, determine whether the Law of Large Numbers holds when $S_n$ is defined as the total number of received packets out of $n$ packets sent. Do a calculation to justify each answer. (Whenever convenient, you can assume below that $n$ is even.)

a) Yes or No: Each packet is sent on a completely different route.

b) Yes or No: The packets are split into $n/2$ pairs of packets. Each pair is sent together on its own route (i.e., different pairs are sent on different routes).

c) Yes or No: The packets are split into 2 groups of $n/2$ packets. All the packets in each group are sent on the same route, and the two groups are sent on different routes.

d) Yes or No: All the packets are sent on one route.

Solution:

a) Yes. Define a set of indicator functions

$$Z_i := 1_{\{\text{Packet } i \text{ is received}\}}, \quad 1 \leq i \leq n.$$  

Since each packet is sent on a different route, $Z_i$’s are i.i.d. Bernoulli random variables with mean $1 - p$ and bounded variance. Since $S_n = \sum_{i=1}^{n} Z_i$, LLN ensures that

$$\lim_{n \to \infty} \mathbb{P}\left( \frac{1}{n} |S_n - \mathbb{E}[S_n]| > \epsilon \right) = 0 \quad \forall \epsilon > 0$$

$$\implies \lim_{n \to \infty} \mathbb{P}\left( \frac{1}{n} |S_n - n(1-p)| > \epsilon \right) = 0 \quad \forall \epsilon > 0.$$  

b) Yes. Suppose that for $1 \leq i \leq n/2$, packets $2i - 1$ and $2i$ are sent on the same route. Define a set of indicator functions

$$Z_i := 1_{\{\text{Packet } 2i \text{ and } 2i-1 \text{ are received}\}}, \quad 1 \leq i \leq \frac{n}{2}.$$  

Since each pair uses a different route, $Z_i$’s are i.i.d. Bernoulli random variables with mean $1 - p$ and bounded variance. Since $S_n = \sum_{i=1}^{n/2} 2Z_i$, LLN ensures that

$$\lim_{n \to \infty} \mathbb{P}\left( \frac{1}{n} |S_n - \mathbb{E}[S_n]| > \epsilon \right) = 0 \quad \forall \epsilon > 0$$

$$\implies \lim_{n \to \infty} \mathbb{P}\left( \frac{1}{n} |S_n - n(1-p)| > \epsilon \right) = 0 \quad \forall \epsilon > 0.$$  

Observing that $\mathbb{E}[S_n] = n(1 - p)$ completes the proof.
c) No. Observe that $S_n$ can only take 3 different values ($S_n = n$, $S_n = n/2$, or $S_n = 0$). For instance, $S_n = n$ when both routes do not fail, which occurs with probability $(1 - p)^2 > 0$, and $S_n = 0$ when both routes fail, which occurs with probability $p^2 > 0$. Consequently, $S_n$ cannot converge to a constant value in probability.

d) No. $S_n$ can only take 2 different values: i) $S_n = n$ when the route works, which occurs with probability $1 - p > 0$, and ii) $S_n = 0$ when the route fails, which occurs with probability $p > 0$. Consequently, $S_n$ cannot converge to a constant value in probability.

4. a) $X_i$’s are i.i.d. draws from the same pdf $f_X$. You know the mean $\mu$ of the pdf $f_X$ but not its variance $\sigma^2$. Can you design an estimator for the variance from the data $X_1, X_2, \ldots$? Show that your estimator converges to $\sigma^2$, and make precise the notion of convergence. (Assume that the variance exists).

b) Suppose now you don’t know even the mean. Can you design an estimator for the variance? Show that your estimator converges to $\sigma^2$, in the same sense as in part (a). (Assume that the variance exists).

Solution:

a) Choose the estimator $\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2$. Each of the random variables $(X_i - \mu)^2$ are independent of each other, and $E[(X_i - \mu)^2] = \sigma^2$. Hence $E[\hat{\sigma}_n^2] = \sigma^2$. Thus we can use the WLLN to say

$$\lim_{n \to \infty} \Pr\left( |\hat{\sigma}_n^2 - \sigma^2| > \epsilon \right) = 0 \quad \forall \ \epsilon > 0$$

Therefore the estimator converges in probability to $\sigma^2$.

b) Since we do not know the mean, we first define $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Then estimate the variance as $\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$. Observe that

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 2 \frac{1}{n} \sum_{i=1}^{n} X_i \bar{X}_n + \frac{1}{n} \sum_{i=1}^{n} \bar{X}_n^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 2 \bar{X}_n^2 + \bar{X}_n^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \bar{X}_n^2$$

Let $W_n = \frac{1}{n} \sum_{i=1}^{n} X_i^2$. Using the law of large numbers,

$$\lim_{n \to \infty} \Pr\left( |W_n - E[X^2]| > \epsilon \right) = 0 \quad \forall \ \epsilon > 0$$
Using the law of large numbers, we also know that

$$\lim_{n \to \infty} \Pr \left( |\bar{X}_n - \mu| > \epsilon \right) = 0 \quad \forall \epsilon > 0$$

Since $\bar{X}_n \to \mu$ and $W_n \to E[X^2]$ in probability as $n \to \infty$, $\tilde{\sigma}_n^2 = W_n - \bar{X}_n^2$ converges in probability to $E[X^2] - \mu^2 = \sigma^2$ as $n \to \infty$.

**Note:** In part b), you cannot use the LLN on $\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$ because it is not the sum of independent random variables.

5. Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be two sequences of random variables. The random variables are all independent. The $A_i$’s are identically distributed, and the $B_i$’s are identically distributed, but the $A_i$’s and $B_i$’s do not have the same distribution. Can you say something about the distribution of $S_n$ for $n$ large, where

$$S_n := \sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i$$

? Make as precise a statement as you can.

**Solution:**

Suppose that the $A_i$’s have mean $\mu_A$ and variance $\sigma_A^2$, and the $B_i$’s have mean $\mu_B$ and variance $\sigma_B^2$. From the Central Limit Theorem, we know that

$$\frac{\sum_{i=1}^{n} A_i - n\mu_A}{\sqrt{n}\sigma_A} \sim N(0, 1)$$

$$\implies \sum_{i=1}^{n} A_i \sim N(n\mu_A, n\sigma_A^2)$$

Similarly, we also know that

$$\sum_{i=1}^{n} B_i \sim N(n\mu_B, n\sigma_B^2)$$

We also know that the sum of two independent Gaussian random variables is also a Gaussian random variable with mean as the sum of means and variance as the sum of variances, i.e.

$$S_n \sim N \left( n(\mu_A + \mu_B), n(\sigma_A^2 + \sigma_B^2) \right)$$