Homework 6

Due: Wed, Nov 11-2020, 11:59pm – Gradescope entry code: 948XVG

Please upload your answers timely to Gradescope. Start a new page for every problem.

1. Exercise 10.5 in text.

   Exercise 10.5  (a) Assume that $X_1 \sim \mathcal{N}(\bar{X}_1, \sigma^2_{X_1})$ and that for each $n \geq 1$, $X_{n+1} = \alpha X_n + W_n$, where $0 < \alpha < 1$, $W_n \sim \mathcal{N}(0, \sigma^2_W)$, and $X_1, W_1, W_2, \ldots$, are independent. Show that for each $n \geq 1$,
   
   $$E[X_n] = \alpha^{n-1}\bar{X}_1, \quad \sigma^2_{X_n} = \frac{(1 - \alpha^{2(n-1)})\sigma^2_W}{1 - \alpha^2} + \alpha^{2(n-1)}\sigma^2_{X_1}. \tag{1}$$

   (b) Show directly, by comparing the equation $\sigma^2_{X_n} = \alpha^2\sigma^2_{X_{n-1}} + \sigma^2_W$ for each two adjoining values of $n$, that $\sigma^2_{X_n}$ moves monotonically from $\sigma^2_{X_1}$ to $\sigma^2_W/(1-\alpha^2)$ as $n \to \infty$.

   (c) Assume that sample values of $Y_1, Y_2, \ldots, Y_n$ are observed, where $Y_n = hX_n + Z_n$ and where $Z_1, Z_2, \ldots, Z_n$ are IID zero-mean Gaussian rvs with variance $\sigma^2_Z$. Assume that $Z_1, \ldots, Z_n, X_1$ are independent and assume $h \geq 0$. Rewrite the recursion for the variance of the estimation error in (10.41) for this special case. Show that if $h/\sigma_Z = 0$, then $\sigma^2_{e_n} = \sigma^2_{X_1}$ for each $n \geq 1$. Hint: Compare the recursion in (b) to that for $\sigma^2_{e_n}$.

   (d) Show from the recursion that $\sigma^2_{e_n}$ is a decreasing function of $h/\sigma_Z$ for each $n \geq 2$. Use this to show that $\sigma^2_{e_n} \leq \sigma^2_{X_1}$ for each $n$. Explain (without equations) why this result must be true.

   (e) Show that the sequence $(\sigma^2_{e_n}; n \geq 1)$ is monotonic in $n$. Hint: Use the same technique as in (b). From this and (d), show that $\lambda = \lim_{n \to \infty} \sigma^2_{e_n}$ exists. Show that the limit satisfies (10.42) (note that (10.42) must have two roots, one positive and one negative, so the limit must be the positive root).

   (f) Show that for each $n \geq 1$, $\sigma^2_{e_n}$ is increasing in $\sigma^2_{W_1}$ and increasing in $\alpha$. Note: This increase with $\alpha$ is surprising, since when $\alpha$ is close to $1$, $X_n$ changes slowly, so we would expect to be able to track $X_n$ well. The problem is that $\lim_n \sigma^2_{e_n} = \sigma^2_W/(1-\alpha^2)$ so the variance of the untracked $X_n$ is increasing without limit as $\alpha$ approaches 1. Part (g) is somewhat messy, but resolves this issue.

   (g) Show that if the recursion is expressed in terms of $\beta = \sigma^2_W/(1-\alpha^2)$ and $\alpha$, then $\lambda$ is decreasing in $\alpha$ for constant $\beta$.

2. a) In Q. 1 (10.5 in text), fix $\bar{X}_1 = 0$, $\sigma^2_Z = 1$, $h = 1$, $\sigma^2_{X_1} = 1$ and $\beta = 1$. Consider 4 possible values for $\alpha$: $\alpha = 0.1, 0.5, 0.9, 0.99$. For each of these values, simulate the system and plot a realization of $\{X_n\}$ and a realization of $\{\hat{X}_n\}$ on the same plot.
Plot them for long enough time that the system reaches steady state. Explain how the plot qualitatively changes as $\alpha$ varies.

b) In Q. 1, compute for general parameter values the impulse response of the LTI system from $\{W_n\}$ to $\{X_n\}$ and the impulse response of the estimator from $\{Y_n\}$ to $\{\hat{X}_n\}$. For the parameter values in part (a), plot the two impulse responses. How do the impulse responses qualitatively change with $\alpha$? Is this consistent with your answer to part (a)?

3. Consider the vector dynamical system:

$$\begin{align*}
X_1 &\sim \mathcal{N}(\bar{X}, K_1) \\
X_n &= AX_{n-1} + BW_n \quad n = 2, 3, \ldots \\
Y_n &= CX_n + DZ_n \quad n = 1, 2, \ldots
\end{align*}$$

where $W_1, \ldots, Z_1, \ldots$, are independent and $W_n \sim \mathcal{N}(0, K_w)$ and $Z_n \sim \mathcal{N}(0, K_z)$.

a) Reformulate the scalar system:

$$\begin{align*}
X_1, X_2 &\sim \mathcal{N}(0, 1) \\
X_n &= X_{n-1} + 0.5X_{n-2} + W_n, \quad n = 3, 4, \ldots \\
Y_n &= X_n + 0.4X_{n-1} + Z_n, \quad n = 2, 3, \ldots
\end{align*}$$

as a vector dynamical system. Here, $X_1$ and $X_2$ are independent and $W_n$’s and $Z_n$’s are independent and $\mathcal{N}(0, 1)$ distributed, and independent of $X_1$ and $X_2$.

b) For the general vector dynamical system, derive the recursion for the Kalman filter estimates $\hat{X}_n(Y_1, Y_2, \ldots Y_n)$ and for the covariance matrices of the errors. Assuming that the error covariance matrices approach a limit as $n \to \infty$, characterize the limit in terms of the solution of an equation.

4. Consider a truck on frictionless, straight rails. Initially, the truck is stationary at location $L_0 = 0$ and moving with velocity $V_0 = 1$, but it is buffeted by random uncontrolled forces. We measure the position of the truck every $\Delta t = 1$ seconds, but these measurements are imprecise; we want to maintain a model of the truck’s location $L_t$ and its velocity $V_t$. Specifically, assuming at time $t = 0$, the initial state of the truck is $L_0 = 0$, $V_0 = 1$. Between $t - 1$ and $t$, the velocity is subject to a constant acceleration $A_{t-1} \sim \mathcal{N}(0, \sigma_a^2)$. Also at time $t$, we take a noisy observation $Y_t = L_t + Z_t$, where $Z_t \sim \mathcal{N}(0, \sigma_z^2)$.

Formulate this problem as a vector Kalman filter estimation problem in the form of Problem 3 above.

a) What is the estimate $\hat{X}_1(y_1)$ at $t = 1$? What is the covariance matrix $K_{\xi_1}$ of the error of this estimator? Give your answer in terms of $y_1, \sigma_a^2, \sigma_z^2$.

b) What is the estimate $\hat{X}_2(y_1, y_2)$ at $t = 2$? What is the covariance matrix of the error of this estimator? Give your answer in terms of $y_2, \sigma_a^2, \sigma_z^2, \hat{X}_1(y_1), K_{\xi_1}$. (You can use the recursion from 3. b) above).