Homework 7

Due: Tues, Nov 17-2020, 11:59pm – Gradescope entry code: 948XVG

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1. a) Are the following processes stationary? For each, compute the mean function, covariance function \( K_X(n_1, n_2) \) and sketch a typical sample path. If a process is stationary, compute also its power spectral density. Each process is defined on \( n = 0, 1, 2, \ldots \).

i. \( X_n = A \sim N(0, 1) \).

ii. \( X_n \) i.i.d. \( N(0, 1) \) random variables.

iii. \( X_n = nA \), where \( A \sim N(0, 1) \).

iv. \( X_n = \sin(n\omega + \Theta) \), where \( \omega \) is a fixed angular frequency, and \( \Theta \) is uniformly distributed between 0 and \( 2\pi \).

b) Let \( A_n \) and \( B_n \) be two independent stationary processes with means \( \mu_A \) and \( \mu_B \) respectively, and with covariance functions \( K_A(\cdot) \) and \( K_B(\cdot) \) respectively. Is the process \( \{C_n = A_n + B_n\} \) stationary? If so, compute the mean, covariance function and power spectral density of \( \{C_n\} \) in terms of the corresponding quantities for \( \{A_n\} \) and \( \{B_n\} \).

**Solution:**

a) Find the solution for the mean and covariance functions, sample sketches and stationarity in the solutions for Section 8.

i. \( K_X(m) = 1 \) for all \( m \). Taking its Fourier transform, \( S_X(f) = \delta(f) \).

ii. \( K_X(m) = 1 \) if \( m = 0 \) or 0 otherwise, i.e. \( K_X(m) = \delta[m] \). Taking its Fourier transform, \( S_X(f) = 1 \).

iii. This process is not stationary.

iv. \( K_X(m) = \frac{1}{2} \cos(m\omega) \). Its Fourier transform is

\[
S_X(f) = \frac{1}{4} \left( \delta(f - \frac{\omega}{2\pi}) + \delta(f + \frac{\omega}{2\pi}) \right).
\]

b) For any \( n, k \), the joint distribution of \( \{C_n, C_{n+1}, \ldots, C_{n+k}\} \) is the product of the joint distribution of \( \{A_n, A_{n+1}, \ldots, A_{n+k}\} \) and the joint distribution of \( \{B_n, B_{n+1}, \ldots, B_{n+k}\} \). If \( \{A_n\} \) and \( \{B_n\} \) are stationary, then these joint distributions are the same for all \( n \), hence \( \{C_n\} \) is also stationary.
Due to linearity of expectation,

\[ \mu_C = \mu_A + \mu_B. \]

Let \( \tilde{A}_n = A_n - \mu_A \) and \( \tilde{B}_n = B_n - \mu_B \).

\[
K_C(m) = \mathbb{E}[(C_0 - \mu_C)(C_m - \mu_C)]
= \mathbb{E}[(\tilde{A}_0 + \tilde{B}_0)(\tilde{A}_m + \tilde{B}_m)]
= \mathbb{E}[\tilde{A}_0 \tilde{A}_m] + \mathbb{E}[\tilde{B}_0 \tilde{B}_m] 
\text{ due to independence}
= K_A(m) + K_B(m)
\]

Due to linearity of the Fourier transform,

\[
S_C(f) = S_A(f) + S_B(f).
\]

2. a) For Q. 2 in HW 6, compute the covariance functions and power spectral density for the processes \( \{X_n\}, \{Y_n\} \) and \( \{\hat{X}_n\} \) for the parameter values specified as in Q. 2(a) in HW 6 and for the 4 different values of \( \alpha \).

b) Comment on how the covariance functions you computed in part (a) depends on \( \alpha \). Are the dependency consistent with your plots in Q. 2 in HW 6?

Solution:

a) i. For \( \alpha < 1 \), these processes are stationary only in the steady state where the variance converges to a constant value. The system dynamics is given by

\[
X_{n+m} = \alpha X_{n+m-1} + W_{n+m-1} = \alpha (\alpha X_{n+m-2} + W_{n+m-2}) + W_{n+m-1}
= \alpha^2 X_{n+m-2} + \alpha W_{n+m-2} + W_{n+m-1}
= \cdots = \alpha^m X_n + \alpha^{m-1} W_n + \cdots + W_{n+m-1}.
\]

In this problem, we set \( \sigma_W^2 = \beta(1 - \alpha^2) = 1 - \alpha^2 \). At steady state, \( \sigma_X^2 = \sigma_W^2/(1 - \alpha^2) = 1. \) Since \( X_n \) is independent of \( W_n, \ldots, W_{n+m-1} \),

\[
K_X(n, n+m) = \mathbb{E}[X_n X_{n+m}]
= \alpha^m \mathbb{E}[X_n^2]
= \alpha^m
\]

Thus, \( K_X(m) = \alpha^{|m|} \). By taking the Fourier transform,

\[
S_X(f) = \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha \cos(2\pi f)}
\]

Alternatively, recall from the solution to Q. 2(b), that the impulse response of the LTI system from \( W_n \) to \( X_n \) is \( h(m) = \alpha^{m-1} u[m-1] \). Since \( W_n \) is an i.i.d.
process, \( K_W(m) = \sigma_W^2 \delta[m] = (1 - \alpha^2) \delta[m] \). Then by passing \( W_n \) through this LTI system,
\[
K_X(m) = (h * h_R * K_Z)(m)
= (1 - \alpha^2) \sum_{n=-\infty}^{\infty} h(n) h_R(m-n)
= (1 - \alpha^2) \sum_{n=-\infty}^{\infty} \alpha^{n-1} u[n-1] \alpha^{n-m-1} u[n-m-1]
= (1 - \alpha^2) \sum_{n=m+1}^{\infty} \alpha^{2n-m-2}
= (1 - \alpha^2) \frac{\alpha^m}{1 - \alpha^2} = \alpha^m,
\]
\[
S_X(f) = |H(f)|^2 S_W(f)
= \left| \frac{e^{-j2\pi f}}{1 - \alpha e^{-j2\pi f}} \right|^2 (1 - \alpha^2)
= \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha \cos(2\pi f)}.
\]

ii. \( Y_n = X_n + Z_n \) where \( Z_n \) is independent of \( X_n \). \( Z_n \) is i.i.d. with \( \sigma_Z^2 = 1 \), so \( K_Z(m) = \delta[m] \). Using the answer from Q. 1(b) above,
\[
K_Y(m) = K_X(m) + K_Z(m)
= \alpha^{|m|} + \delta[m],
\]
\[
S_Y(f) = S_X(f) + S_Z(f)
= \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha \cos(2\pi f)} + 1.
\]

iii. Recall from the solution to Q. 2(b) in HW 6, that \( \sigma^2_{\xi_n} \) converges to a value \( \sigma^2_{\xi_\infty} \) which satisfies
\[
\alpha^2 \sigma^4_{\xi_\infty} + 2(1 - \alpha^2) \sigma^2_{\xi_\infty} - (1 - \alpha^2) = 0
\]
\[
\Rightarrow \sigma^2_{\xi_\infty} = \frac{\sqrt{1 - \alpha^2} - (1 - \alpha^2)}{\alpha^2}.
\]

The steady state recursion is given by
\[
\hat{X}_n(y^n_1) = \alpha (1 - \gamma_\infty) \hat{X}_{n-1}(y^{n-1}_1) + \gamma_\infty y_n
\]
where \( \gamma_\infty = \frac{\alpha^2 \sigma^2_{\xi_\infty} + 1 - \alpha^2}{\alpha^2 \sigma^2_{\xi_\infty} + 1 - \alpha^2 + 1} = \frac{\sqrt{1 - \alpha^2}}{1 + \sqrt{1 - \alpha^2}}. \)
Expanding the recursion, we have
\[
\hat{X}_{n+m} = \alpha(1 - \gamma_\infty)\hat{X}_{n+m-1} + \gamma_\infty Y_{n+m}
\]
\[
= \alpha(1 - \gamma_\infty)\left(\alpha(1 - \gamma_\infty)\hat{X}_{n+m-2} + \gamma_\infty Y_{n+m-1}\right) + \gamma_\infty Y_{n+m}
\]
\[
= \alpha^2(1 - \gamma_\infty)^2\hat{X}_{n+m-2} + \alpha(1 - \gamma_\infty)\gamma_\infty Y_{n+m-1} + \gamma_\infty Y_{n+m}
\]
\[
= \cdots = \alpha^{m}(1 - \gamma_\infty)^m\hat{X}_{n} + \alpha^{m-1}(1 - \gamma_\infty)^{m-1}\gamma_\infty Y_{n+1} + \cdots + \gamma_\infty Y_{n+m}.
\]
Thus,
\[
K_{\hat{X}}(n, n + m) = \mathbb{E}[\hat{X}_n\hat{X}_{n+m}]
\]
\[
= \alpha^m(1 - \gamma_\infty)^m\mathbb{E}[\hat{X}_n^2] + \alpha^{m-1}(1 - \gamma_\infty)^{m-1}\gamma_\infty\mathbb{E}[\hat{X}_n Y_{n+1}] + \cdots + \gamma_\infty^m\mathbb{E}[\hat{X}_n Y_{n+m}]
\]
\[
= \alpha^m(1 - \gamma_\infty)^m\mathbb{E}[\hat{X}_n^2] + \alpha^{m-1}(1 - \gamma_\infty)^{m-1}\gamma_\infty^2\mathbb{E}[\hat{X}_n^2] + \cdots + \gamma_\infty^m\alpha^m\mathbb{E}[\hat{X}_n^2]
\]
\[
= \alpha^{m}(1 - \gamma_\infty)^m\mathbb{E}[\hat{X}_n^2] + \alpha^{m} \gamma_\infty \frac{1}{1 - (1 - \gamma_\infty)^m} \mathbb{E}[\hat{X}_n^2]
\]
\[
= \alpha^m(1 - \sigma_\xi^2).
\]
The third line above is because
\[
\mathbb{E}\[Y_{n+1}\hat{X}_n] = \mathbb{E}[(\alpha^i X_n)\hat{X}_n] = \mathbb{E}\[\alpha^i \hat{X}_n^2] = \alpha^i \mathbb{E}\[\hat{X}_n^2].
\]
Thus, \(K_{\hat{X}}(m) = \alpha^{|m|}(1 - \sigma_\xi^2)\). Taking the Fourier transform, we get
\[
S_{\hat{X}}(f) = \frac{(1 - \alpha^2)(1 - \sigma_\xi^2)}{1 + \alpha^2 - 2\alpha \cos(2\pi f)}.
\]
However, the easier way to solve this is using the impulse response of the LTI system from \(Y_n\) to \(\hat{X}_n\), \(h(m) = \gamma_\infty \alpha^m(1 - \gamma_\infty)^m u[m]\). We have already calculated that \(K_Y(m) = \alpha^{|m|} + \delta[m]\). Then by passing \(Y_n\) through this LTI system,
\[
S_{\hat{X}}(f) = |H(f)|^2 S_Y(f)
\]
\[
= \left| \frac{\gamma_\infty}{1 - \alpha(1 - \gamma_\infty)e^{-j2\pi f}} \right|^2 \left( \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha \cos(2\pi f)} + 1 \right)
\]
\[
= \frac{\gamma_\infty^2}{1 + \alpha^2(1 - \gamma_\infty)^2 - 2\alpha(1 - \gamma_\infty) \cos(2\pi f)} \cdot \frac{2 - 2\alpha \cos(2\pi f)}{1 + \alpha^2 - 2\alpha \cos(2\pi f)}
\]
By substituting the value of \(\gamma_\infty\),
\[
S_{\hat{X}}(f) = \frac{1 - \alpha^2}{(1 + \sqrt{1 - \alpha^2})^2 + \alpha^2 - 2\alpha(1 + \sqrt{1 - \alpha^2}) \cos(2\pi f)} \cdot \frac{2 - 2\alpha \cos(2\pi f)}{1 + \alpha^2 - 2\alpha \cos(2\pi f)}
\]
\[
= \frac{1 - \alpha^2}{(1 + \sqrt{1 - \alpha^2})(2 - 2\alpha \cos(2\pi f))} \cdot \frac{2 - 2\alpha \cos(2\pi f)}{1 + \alpha^2 - 2\alpha \cos(2\pi f)}
\]
\[
= \frac{1}{1 + \sqrt{1 - \alpha^2}} \cdot \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha \cos(2\pi f)}
\]
We can then take the inverse Fourier transform to get

\[ K_X(m) = \frac{\alpha^{|m|}}{1 + \sqrt{1 - \alpha^2}} = (1 - \sigma_\xi^2)\alpha^{|m|} \]

Note: Useful Fourier transforms used in this solution:

\[ x(m) = a^m u[m] \quad \leftrightarrow \quad X(f) = \frac{1}{1 - ae^{-j2\pi f}} \quad \text{for} \ |a| < 1 \]
\[ x(m) = a^{|m|} \quad \leftrightarrow \quad X(f) = \frac{1 - a^2}{1 + a^2 - 2a \cos(2\pi f)} \quad \text{for} \ |a| < 1 \]
\[ x(m) = \delta[m] \quad \leftrightarrow \quad X(f) = 1 \]
\[ y(m) = x(m - m_0) \quad \leftrightarrow \quad Y(f) = X(f) e^{-j2\pi fm_0} \]

b) The covariance functions decay as \( \alpha^{|m|} \). As \( \alpha \) becomes close to 0, the covariance function becomes an impulse, so adjacent values are fully uncorrelated. As \( \alpha \) becomes close to 1, the covariance functions become a constant, so there is very high correlation between adjacent samples.

In the plots of Q. 2(b) in HW 6, adjacent values of \( X_n \) and of \( \hat{X}_n \) are highly uncorrelated when \( \alpha \) is small and much more correlated when \( \alpha \) is close to 1. Hence our conclusions are consistent with those plots.

3. a) Suppose there is a zero-mean stationary random process \( \{X_n : n \geq 0\} \) but you don’t know its covariance function \( K_X(m) \). You observe a sample path of the process up to time \( N \). Find a reasonable estimator \( \hat{K}^{(N)}(m) \) of the covariance function \( K_X(m) \) for every \( m \).

b) Extend part (a) to the non-zero mean case.

c) The Madden Julian oscillation (MJO) is the largest element of the intraseasonal variability in the tropical atmosphere. It is a traveling pattern that propagates eastward, and each cycle lasts approximately 30-60 days. The MJO is most obvious in the variation of outgoing longwave radiation (OLR), a satellite-derived measure of tropical convection and rainfall.

The data package at [http://ee278.stanford.edu/homeworks/hw7_data.mat](http://ee278.stanford.edu/homeworks/hw7_data.mat) consists of one time series \( \{X_n\} \) of OLR in the Indian ocean that contains 1752 points. Each point in the time series records a 5-day average of OLR over the region specified, from the beginning of 1979 through the end of 2002. Suppose that the time series is generated by a stationary process.

Estimate the covariance function of the time series and plot it for time lags 0 through 30. Describe briefly on any cyclic characteristics that the covariance function might present. What can you say about the periodicity of the MJO?

Solution:
a) You can estimate the covariance function as

$$\hat{K}_X^{(N)}(m) = \frac{1}{N - m} \sum_{i=1}^{N-m} X(i)X(i + m)$$

for every $m \geq 0$. For $m < 0$, you can set $\hat{K}_X^{(N)}(m) = \hat{K}_X^{(N)}(-m)$.

b) First estimate the mean of the process as

$$\bar{X}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} X(i)$$

Then estimate the covariance function as

$$\hat{K}_X^{(N)}(m) = \frac{1}{N - m} \sum_{i=1}^{N-m} \left( X(i) - \bar{X}^{(N)} \right) \left( X(i + m) - \bar{X}^{(N)} \right)$$

for every $m \geq 0$. For $m < 0$, you can set $\hat{K}_X^{(N)}(m) = \hat{K}_X^{(N)}(-m)$.

c) We can estimate the covariance function using the estimator defined above. The estimates are plotted in Fig. [1]. From the plot, we can see that the fluctuation of the covariance function seems to exhibit some periodicity, i.e. the period of peaks is approximately 8 - 10 points, corresponding to the cycle of 40 - 50 days in the MJO data.

Practically, the uncertainty and noise often make it difficult to spot oscillatory behavior in a random signal, even if such behavior is expected. The covariance function of a random periodic signal has the same cyclic characteristics as the signal itself, and can be estimated in a more robust fashion. Thus, the covariance function is a more robust measure for detecting the presence of periodicity and determining their durations.
Figure 1: Covariance Function