1. **Poisson process probabilities.** Consider a Poisson process with intensity $\lambda > 0$.
   a. Find the probability that there is (exactly) one arrival in each of the three intervals $[0, 1]$, $[1, 2]$, and $[2, 3]$.
   b. Find the probability that there are two arrivals in the interval $[0, 2]$ and two arrivals in the interval $[1, 3]$. (Note: your answer should be larger than the answer to part (a)).
   c. Find the probability that there are two arrivals in the interval $[1, 2]$ given that there are two arrivals in $[0, 2]$ and two arrivals in $[1, 3]$.

2. **Autocorrelation functions.** Find the autocorrelation functions of
   a. random binary waveform $X(t)$:
      In a digital communication channel, the symbol “1” is represented by the fixed duration rectangular pulse
      \[
g(t) = \begin{cases} 
1 & \text{for } 0 \leq t < T \\
0 & \text{otherwise}
\end{cases}
\]
      and the symbol “0” is represented by $-g(t)$. The data transmitted over the channel is represented by the random process
      \[
X(t) = \sum_{k=0}^{\infty} A_k g(t - kT), \quad t \geq 0,
\]
      where the data $A_0, A_1, A_2, \ldots$ is an i.i.d. random sequence with
      \[
A_i = \begin{cases} 
+1 & \text{with probability } \frac{1}{2} \\
-1 & \text{with probability } \frac{1}{2}
\end{cases}
\]
   b. moving average process $Y_n$:
      Let $\{X_n : n \geq 1\}$ be a discrete-time white Gaussian noise process, that is, $X_1, X_2, X_3 \ldots$ are i.i.d. random variables with $X_n \sim \mathcal{N}(0, N)$. Consider the moving-average process $\{Y_n : n \geq 2\}$ defined by
      \[
Y_n = \frac{2}{3}X_{n-1} + \frac{1}{3}X_{n-2}, \quad n \geq 2.
\]
      with $X_0$ defined to be 0.
3. **Stationary Gauss-Markov process.** Consider the following variation on the Gauss-Markov process:

\[
X_0 \sim \mathcal{N}(0, a) \\
X_n = \frac{1}{2}X_{n-1} + Z_n, \quad n \geq 1,
\]

where \(Z_1, Z_2, Z_3, \ldots\) are i.i.d. \(\mathcal{N}(0,1)\), independent of \(X_0\).

a. Find the mean and autocorrelation functions of \(X_n\).

b. Find \(a\) such that \(X_n\) is wide sense stationary.

4. **Sawtooth process.** Let \(X(t) = g(t-T)\), where \(g(t)\) is the periodic triangular waveform shown in Figure 1 and the delay \(T\) is a random variable with \(T \sim U[0,1]\).

![Figure 1: Periodic triangular waveform](image)

Is \(X(t)\) a strict-sense stationary random process? Justify your answer.

5. **Windowed Poisson process.** Let \(N(t)\), for \(t \geq 0\), be a Poisson process with rate \(\lambda > 0\), and let \(X(t)\), for \(t \geq 0\), be defined by \(X(t) = N(t+1) - N(t)\). Thus, \(X(t)\) is the number of events of \(N\) during the time window \((t, t+1]\).

a. Sketch a typical sample path of \(N\), and the corresponding sample path of \(X\).

b. Find the mean function \(\mu_X(t)\), for \(t \geq 0\) and the autocorrelation function \(R_X(t_1, t_2)\) for \(t_1, t_2 \geq 0\). Express your answer in a simple form.

c. Is \(X\) a Markov process? Why or why not?

6. **Modified telegraph process.** Let \(X(t), Y(t), \) and \(W(t)\) be independent random processes; \(X(t)\) and \(Y(t)\) are zero-mean stationary Gaussian processes with \(R_X(\tau) = R_Y(\tau) = e^{-|\tau|}\). \(W(t)\) is the random telegraph process,

\[
W(t) = A(-1)^{N(t)},
\]

where \(N(t)\) is a Poisson process with parameter \(\lambda\), and the random variable

\[
A = \begin{cases} 
1 & \text{with probability 0.5} \\
-1 & \text{with probability 0.5.}
\end{cases}
\]
A and \( N(t) \) are independent. Now define the new process \( Z(t) \) as

\[
Z(t) = \begin{cases} 
X(t) & \text{if } W(t) = 1 \\
Y(t) & \text{if } W(t) = -1.
\end{cases}
\]

a. Find the first order distribution of \( Z(t) \).
b. Is \( Z(t) \) a Gaussian random process? Justify your answer.
c. Is \( Z(t) \) WSS? Justify your answer.

7. **Generating a random process with a prescribed PSD.** The power spectral density \( S_X(f) \) of every WSS process is real, even, and nonnegative. In this problem you will show that, conversely, if \( S(f) \) is a real, even, nonnegative function such that \( \int_{-\infty}^{\infty} S(f) df < \infty \), then \( S(f) \) is the PSD for some WSS random process. Let us consider the case that

\[
\int_{-\infty}^{\infty} S(f) df = 1.
\]

Define the random process

\[
X(t) = \cos(2\pi F t + \Theta),
\]

where \( F \sim S(f) \) and \( \Theta \sim U[0, 2\pi] \) are independent.

a. Show that \( X(t) \) is WSS.
b. Find the power spectral density of \( X(t) \). Interpret the result.