Homework #5
Due Wednesday, May 4

Please drop the assignment in the HOMEWORK In box in the EE 278 drawer of the class file cabinet on the 2nd floor of the Packard Building before 5pm.

1. Prediction. Let $|\alpha| < 1$, and let $\mathbf{X}$ be a zero-mean random vector with covariance matrix

$$
\Sigma_{\mathbf{X}} = \begin{bmatrix}
1 & \alpha & \alpha^2 & \cdots & \alpha^{n-2} & \alpha^{n-1} \\
\alpha & 1 & \alpha & \cdots & \alpha^{n-3} & \alpha^{n-2} \\
\alpha^2 & \alpha & 1 & \cdots & \alpha^{n-4} & \alpha^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha^{n-2} & \alpha^{n-3} & \alpha^{n-4} & \cdots & 1 & \alpha \\
\alpha^{n-1} & \alpha^{n-2} & \alpha^{n-3} & \cdots & \alpha & 1 
\end{bmatrix}.
$$

Find the best linear MSE estimate of $X_n$ given $X_1, X_2, \ldots, X_{n-1}$ and its MSE.

2. Noise cancellation. A classic problem in statistical signal processing involves estimating a weak signal (e.g., the heart beat of a fetus) in the presence of a strong interference (the heart beat of its mother) by making two observations—one with the weak signal present and one without (by placing one microphone on the mother’s belly and another close to her heart). The observations can then be combined to estimate the weak signal by “canceling out” the interference. The following is a simple version of this application.

Let the weak signal $X$ be a random variable with mean $\mu$ and variance $P$. Let the observations be $Y_1 = X + Z_1$ and $Y_2 = Z_1 + Z_2$, where $Z_1$ is the strong interference and $Z_2$ is measurement noise. Assume that $Z_1$ and $Z_2$ are zero mean with variances $N_1$ and $N_2$, respectively. Further assume that $X, Z_1,$ and $Z_2$ are uncorrelated. Find the best linear MSE estimate of $X$ given $Y_1$ and $Y_2$ and the corresponding MSE. Interpret the results.

3. Additive nonwhite Gaussian noise channel. Let $Y_i = X + Z_i$ for $i = 1, 2, \ldots, n$ be $n$ observations of a signal $X \sim \mathcal{N}(0, P)$. The additive noise random variables $Z_1, Z_2, \ldots, Z_n$ are zero mean jointly Gaussian random variables that are independent of $X$ and have correlation $\text{E}(Z_iZ_j) = N \cdot 2^{-|i-j|}$ for $1 \leq i, j \leq n$. Find the best MSE estimate of $X$ given $Y_1, Y_2, \ldots, Y_n$, and its MSE. Hint: the coefficients for the best estimate are of the form $\mathbf{h}^T = [a \ b \ b \ \cdots \ b \ b \ a]$.

4. An innovations sequence and its applications (due to B. Hajek). Let $[Y_1 \ Y_2 \ Y_3 \ X]^T$ be a zero-mean random vector with covariance matrix

$$
\begin{bmatrix}
1 & 0.5 & 0.5 & 0 \\
0.5 & 1 & 0.5 & 0.25 \\
0.5 & 0.5 & 1 & 0.25 \\
0 & 0.25 & 0.25 & 1 
\end{bmatrix}.
$$

a. Let $\tilde{\mathbf{Y}} = [\tilde{Y}_1 \ \tilde{Y}_2 \ \tilde{Y}_3]^T$ be the innovations sequence of $\mathbf{Y} = [Y_1 \ Y_2 \ Y_3]^T$. Find the matrix $A$ such that

$$
\tilde{\mathbf{Y}} = A\mathbf{Y}.
$$
b. Find the covariance matrix of $\tilde{Y}$ and the crosscovariance matrix of $X$ and $\tilde{Y}$.

c. Find the constants $a$, $b$, and $c$ that minimize $\mathbb{E}[(X - a\tilde{Y}_1 - b\tilde{Y}_2 - c\tilde{Y}_3)^2]$.

5. The filtering version of the Kalman filter. Derive the Kalman filter update equations for estimating $\hat{X}_{i|i}$ and its mean squared error $\sigma_{i|i}$.

6. Gaussian noise filtering. In this problem, we apply Gaussian vector estimation in image reconstruction. In particular, we examine the MNIST hand-written number dataset, where we recover the hand written 1’s corrupted by Gaussian noise.

Each image is of size 28 by 28, and is represented by $Y_i$, $i = 1, \ldots, n$, a length 768 column vector after concatenating the columns. The pixels are represented as uint8 integers ranging from 0 to 255 (0 indicates black and 255 is white). Each image vector $Y_i$ is then corrupted by iid Gaussian vectors $N_i \in \mathbb{R}^{768}$ with covariance matrix $\Sigma$, resulting in $X_i = Y_i + N_i$. We would now like to recover $Y_i$’s from $X_i$’s.

a. Estimate the covariance matrix $\Sigma_{YY}$ of the image vectors by completing the corresponding code blocks. You may find numpy.cov() useful.

b. Compute the covariance matrices $\Sigma_{XX}$ and $\Sigma_{XY}$ and complete the code blocks.

c. Complete the denoise method in the code block. Test the denoising algorithm when iid Gaussian noise ($\Sigma = \sigma^2 I_{768 \times 768}$) is added to the image vectors. Use the method add_noise() provided.

d. Test the denoising function on colored noise with $\Sigma$ randomly generated. The code to generate random covariance structure has been provided.

e. Comment on the denoising results with different noise variances. Visualize some of results using the provided show() method and attach the results to your submission.