Assessment 1

Due: Thursday – 4pm, Gradescope entry code: 948XVG

- Please sign the honor code set up on Gradescope.
- Please upload your answers to Gradescope before 4pm.
- Start a new page for every problem.
- The only allowable aids are a double-sided sheet of notes and a calculator.
- Justify all our answers except for Question 1.
- Questions are weighted differently. The total number of points is 104.

Good luck!
1. (15 points) Which of these matrices are covariance matrices? No justification needed.

a) \[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

b) \[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\]

c) \[
\begin{bmatrix}
1 & -1 \\
1 & 1 \\
\end{bmatrix}
\]

d) \[
\begin{bmatrix}
1 & -1 \\
-1 & 1 \\
\end{bmatrix}
\]

e) \[
\begin{bmatrix}
1 & 2 \\
2 & 1 \\
\end{bmatrix}
\]
2. (30 points) A fraction $f$ of the U.S. voters will vote for Trump in the 2020 election. You would like to estimate $f$ via polling.

a) (10 points) Suppose you randomly poll $n$ voters. The voters are chosen uniformly and independently from the entire U.S. population of voters. Can you construct an estimator for $f$ such that the estimate converges to $f$ as $n \to \infty$? Make precise mathematically what your notion of convergence means.

b) (10 points) Find an $n$ to guarantee that your estimate is within $\pm 0.03$ of the true fraction $f$ with probability at least 95%. Your answer should be an actual number.

c) (10 points) The country is divided into red states and blue states. Now suppose you poll randomly $n/2$ voters from the red states and randomly $n/2$ voters from the blue states. With only this data and no other information, can you construct an estimator for $f$ such that the estimate converges to $f$? If yes, provide such an estimator. If not, state what additional information you need.
3. (25 points) Let $X$ be a Gaussian random vector with mean $\mu$ and covariance matrix $K$ given by

$$\mu = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$ 

a) (5 points) What is the distribution of $X_1$?

b) (7 points) What is the distribution of $X_1 + X_2$?

c) (7 points) What is the distribution of $X_3$ given $X_1 = x_1$ and $X_2 = x_2$?

d) (6 points) We wish to project $X$ onto 1 dimension. Find the unit-norm vector along which the projection of $X$ has the largest variance.
4. (34 points) Let the signal \( S \) be a random variable defined as follows:

\[
S = \begin{cases} 
0 & \text{with probability } \frac{1}{2} \\
+1 & \text{with probability } \frac{1}{2}
\end{cases}
\]

The signal is sent over a channel with additive exponential noise \( Z_1 \), i.e., \( Z_1 \) is an exponential random variable with pdf with parameter \( \lambda > 0 \):

\[
f_{Z_1}(z) = \begin{cases} 
0 & z < 0 \\
\lambda e^{-\lambda z} & z \geq 0
\end{cases}
\]

The signal \( S \) and the noise \( Z_1 \) are assumed to be independent and the channel output is their sum \( Y_1 = S + Z_1 \).

a) (6 points) Find \( f_{Y_1|S}(y|s) \) for \( s = 0, +1 \). Sketch the conditional pdfs on the same graph.

b) (8 points) Find the optimal decoding rule for deciding whether \( S \) is 0 or +1. Give your answer in terms of ranges of values of \( Y_1 \).

c) (6 points) Find the probability of decoding error of your optimal rule in terms of \( \lambda \).

d) Suppose now there is an additional observation \( Y_2 = S + Z_2 \), where \( Z_2 \) has the same distribution as \( Z_1 \), and \( S, Z_1, Z_2 \) are mutually independent.

i. (8 points) Find the optimal decoding rule for \( S \) given \( Y_1 \) and \( Y_2 \).

ii. (6 points) What is the probability of decoding error of this rule? Is there any improvement over the one when there is a single observation \( Y_1 \)?