Sample Midterm Problems

The following are old midterm problems. The midterm will cover the material in Lecture notes 1, 2, 3, and 4 up to page 4-11, and homeworks 1–4 (including extra problems).

1. Inequalities. Label each of the following statements with =, \( \leq \), \( \geq \), or None. Label a statement with = if equality always holds. Label a statement with \( \geq \) or \( \leq \) if the corresponding inequality holds in general and strict inequality holds sometimes. If no such equality or inequality holds in general, label the statement as None. Justify your answers.
   a. \( E(X_1 X_2 | X_3) \) vs. \( E(X_1 | X_3) E(X_2 | X_3) \) if \( X_1 \) and \( X_2 \) are independent.
   b. \( E[\text{Var}(X | Y, Z)] \) vs. \( E[\text{Var}(X | Y)] \).
   c. \( E[\text{Var}(X | Y)] \) vs. \( E[\text{Var}(X | g(Y))] \). (Hint: Use the result of part (b).)
   d. \( E Z \left[ E(X^2 | Z) \right] \) vs. \( E \left[ E Z(\text{Cov}(X, Y | Z)) \right]^2 \).
   e. \( E \left( \log(2 + \sqrt{X}) \right) \) vs. 1 if \( X \geq 0 \) and \( E(X) \leq 1 \).
   f. \( P\{XY^2 > 16\} \) vs. 1/8 if \( E(X^4) = E(Y^4) = 2 \).

2. Random interval length. Let random variable \( X \sim U[0, 1] \). Then \( X \) divides the unit interval into two subintervals \([0, X]\) and \((X, 1]\) and one of these subintervals is picked in the following way. Let the random variable \( Y \sim U[0, 1] \) be independent of \( X \) and pick the subinterval that \( Y \) falls into. Define the length of the subinterval chosen as \( L \). Thus \( L = X \) if \( Y \leq X \) and \( L = 1 - X \) if \( Y > X \).
   a. Find the probability density function of \( L \).
   b. Suppose that you do not know \( X \) or \( L \) but you observe \( Y \). What is the minimum MSE estimate of \( L \) given \( Y \)? Hint: this can be solved without finding any pdfs. You need only note that \( L \) is a function of \( X \) and \( Y \).

3. Convergence limit uniqueness.
   Prove that if \( X_n \xrightarrow{\ast} n \to \infty Y \), \( X_n \xrightarrow{\ast} n \to \infty X \) where \( \ast \) is a.s, p, m.s., then \( P\{X = Y\} = 1 \)

4. Gaussian Multiple access channel. Consider a vector additive Gaussian noise multiple access channel with two senders and a single receiver. Sender 1 sends \( X_1 = [X_{11} X_{12}]^T \) and Sender 2 sends \( X_2 = [X_{21} X_{22}]^T \). The additive noise is \( Z = [Z_1 Z_2]^T \), and the output is \( Y = X_1 + X_2 + Z \). The joint pmf of the input signals is as follows:
   \[
   \begin{bmatrix}
   X_{11} & X_{21} \\
   X_{12} & X_{22}
   \end{bmatrix}
   = \begin{cases}
   \begin{bmatrix}
   +\sqrt{P} \\
   -\sqrt{P}
   \end{bmatrix}, & \text{with probability 0.5} \\
   \begin{bmatrix}
   0 & -\sqrt{P} \\
   0 & +\sqrt{P}
   \end{bmatrix}, & \text{with probability 0.5}
   \end{cases}
   \]
Thus only one sender is active, i.e., transmits a non-zero signal. The noise components \( Z_1 \) and \( Z_2 \) are independent \( \mathcal{N}(0, N) \), and are independent of the input signals.

a. Specify the optimal rule \( d(y_1, y_2) \) for deciding whether Sender 1 or Sender 2 is active. Your answer should be in terms of explicit decision regions in the \((y_1, y_2)\) plane. (Hint: Define the r.v. \( S = 1 \) if sender 1 transmits and 2 if sender 2 transmits.)

b. Find an expression for the minimum probability of error \( P_e \) in terms only of \( P, N \), and the \( Q \) function. Simplify the form of your answer as much as possible.

5. **Gaussian Random Vector.**

Let \([X \ Y \ Z]^T\) be a zero mean Gaussian random vector with covariance matrix

\[
\Sigma = \begin{bmatrix}
a & b_1 & 0 \\
b_1 & a & b_2 \\
0 & b_2 & a
\end{bmatrix},
\]

where the \( a \) and \( b \) parameters are real-valued NON-ZERO constants.

a. Label each of the following statements as TRUE, FALSE, or NEITHER. Label a statement as TRUE if it always holds. Label a statement as FALSE if it never holds. Label a statement as NEITHER if it may or may not hold. Justify your answers.

- i. \( a > 0 \).
- ii. \( b_1 > 0 \).
- iii. \( a^2 - b_1^2 - b_2^2 > 0 \).
- iv. \( X \) and \( Z \) are independent.
- v. \( X \) and \( Z \) are conditionally independent given \( Y \).

b. Let \( \hat{X} \) be the best MSE estimate of \( X \) given \( Y \). Specify the joint pdf of \( \hat{X} \) and \( Z \). Your answer should be in terms only of the \( a \) and \( b \) parameters.